A Unified Approach to Variational Derivatives of Modified Gravitational Actions

Ahmet Baykal*

Department of Physics, Faculty of Science and Letters, Niğde University, 51240, Niğde, Turkey

Özgür Delice†

Department of Physics, Faculty of Science and Letters, Marmara University, 34722, Istanbul, Turkey

Our main aim in this paper is to promote the coframe variational method as a unified approach to derive field equations for any given gravitational action containing the algebraic functions of the scalars constructed from the Riemann curvature tensor and its contractions. We are able to derive a master equation which expresses the variational derivatives of the generalized gravitational actions in terms of the variational derivatives of its constituent curvature scalars. Using the Lagrange multiplier method relative to an orthonormal coframe, we investigate the variational procedures for modified gravitational Lagrangian densities in spacetime dimensions $n \geqslant 3$. We study well-known gravitational actions such as those involving the Gauss-Bonnet and Ricci-squared, Kretschmann scalar, Weyl-squared terms and their algebraic generalizations similar to generic $f(R)$ theories and the algebraic generalization of sixth order gravitational Lagrangians. We put forth a new model involving the gravitational Chern-Simons term and also give three dimensional New massive gravity equations in a new form in terms of the Cotton 2-form.

I. INTRODUCTION

Although General Relativity theory is an experimentally well-tested and successful theory [1], the search for alternative or modified gravity theories emerged immediately after the formulation of General Relativity (GR). The motivation of introducing such theories varies considerably. For example, in some of the earlier theories, such as those introduced by Kretschmann [2], Weyl [3] and Eddington [4], Lagrangians containing higher order curvature scalars were considered in order to understand mathematical and physical machinery of Einstein's theory and to compare their predictions with different and more generalized theories. In another important route, in the same era, Kaluza and Klein showed that it is possible to unify gravity with electromagnetism from pure geometric Einstein-Hilbert (EH) action but employing an extra compact dimension [5]. Although this theory is not a viable theory, its main idea of employing extra dimensions for the unification survive in modern theories such as String/M theory [6]. In these theories, terms involving higher order curvature scalars arise naturally in gravitational sector of their action [7]. It is also known that, in order to incorporate quantum effects to gravity, one might employ Wheeler-DeWitt effective action formalism [8], which leads to higher order curvature corrections as well [9]. Hence, the studies considering unification as well as quantization of gravity require modifications of some sort in the gravitational Lagrangian.

Another motivation for considering modified gravitational actions comes from the dynamics of the universe. The results of data accumulated on cosmology and astrophysics, in recent years, lead to interesting observations, for example, that the universe is in the epoch of accelerating expansion driven by the dark energy and most of the matter content of the universe is dark [10]. Although the ongoing and future particle physics experiments may explain the physical content of dark matter and dark energy, the possibility that these observations suggests that GR should be modified is also an interesting and attractive idea. Hence, in recent years, the old idea of replacing R in EH action with $f(R)$ [11–15], has been studied extensively in this context. For more information as well as recent developments and their applications to the physically relevant models of $f(R)$ theories, see one of the excellent reviews [16, 17, 19] and references therein. The problem of this approach is that, can these modified gravity theories explain those observational phenomena without bringing out more problems to the area, such as instabilities, and can be compatible with laboratory and observational astrophysical data. Actually, one should regard these attempts as toy models, which might help to understand the gravitational aspects of nature by modeling and studying geometric gravity theories. It is preferable to start investigating modified gravitational models by starting from action scalar for which the metric field

---

*Electronic address: abaykal@nigde.edu.tr
†Electronic address: ozgur.delice@marmara.edu.tr

equations derived from an action via a well-defined variational procedure. There are distinct variational procedures that lead to distinct field equations. In so-called Palatini formalism, the metric tensor and connection are treated as independent variables of the theory and the relation between them is derived as a consequence. In contrast, in the metric formalism, the metric tensor is assumed to be the only field variable determining the spacetime geometry and the field equations for it follows from the variational derivatives of the gravitational action with respect to the components of the metric tensor. Except for the Einstein-Hilbert Lagrangian the metric field equations that follow from these two procedure, as we shall explicitly show below, are not the same. In addition, in the metric approach, the connection (Riemannian or Levi-Civita connection) is derived uniquely from the derivatives of the metric tensor, whereas in the Palatini approach, the connection may be non-Riemannian. This has important consequences on the motions of test particles since a connection defines parallel transport and therefore determine the geodesic structure of the spacetime.

The organization of our paper can be outlined as follows. The paper can roughly be divided into three parts. The first part deals with the constrained variational procedure and Lagrange multiplier technique that will be used in the rest of the paper. The second part is related to the algebraic generalization of modified gravitational actions. In this part, the variational derivative of the well-known $f(R)$ theory in both metric and Palatini approaches and the equivalence of these actions to the Brans-Dicke (BD) type scalar tensor theories are discussed. In the subsequent sections we calculate the variational derivative of the actions which consist of functions of the terms such as Gauss-Bonnet, Ricci-squared or Weyl-squared terms and their algebraic generalizations. Some of the results related to the algebraic generalization of the quadratic curvature terms are new. Various important inter-relationships between the Lagrangians in three and four dimensions are indicated. Finally, the variation of terms involving derivatives of the curvature scalars, which lead to sixth order theories, as well as their algebraic generalizations are discussed. The third part is the original part and it consist of last two sections. These sections can be considered as applications of the general formulae obtained in the previous parts. In the first of these sections, namely section V, a new model for the Chern-Simons gravity in four dimensions is derived whereas the last section is devoted to the study of field equations of three dimensional general massive gravity models and in particular Lagrange multipliers of general massive gravity theory. Our notation and conventions are briefly explained in the Appendix.

II. CONSTRAINED VARIATIONAL DERIVATIVES

We find it convenient to start our discussion in the next section with the general definitions and the remarks about the variational procedure we shall use throughout the paper. The field equations of Einstein's General Theory of Relativity can be derived from Einstein-Hilbert (EH) action

$$I_{tot} = \int_U R * 1 + I_m, \tag{1}$$

via a variational procedure, where the gravitational Lagrangian is given by $\mathcal{L}_{EH} = R * 1$ and R is the Ricci scalar constructed from the Riemann curvature tensor and $*1$ is the invariant volume element. The action for non-gravitational fields coupling to gravity is denoted by I_m. It turns out that any theory which claims to generalize or modify Einstein's GR must include classically well-tested GR as a special case and reduce to GR (and hence Newtonian gravity) in the weak gravity regime [20].

In our work, we shall not address problems arising from certain modified gravitational Lagrangians or their coupling to matter or the phenomenological implications of these theories but scrutinize methods of the derivation of field equations that follow from a given modified gravitational action. The dimensionality of spacetime turns out to be a crucial parameter for the gravitational models, hence we shall consider some popular three dimensional cases after we derive general formulae.

Relative to an orthonormal coframe, metric variational derivatives are induced by coframe variational derivatives and that the vanishing torsion condition can in fact be implemented into the coframe variation either by introducing Lagrange multipliers or directly incorporating this condition into the variation by using the variational derivative $\delta\Theta^\alpha = 0$, which relates the variational derivatives of the connection and coframe 1-forms. Using either of the methods, one ends up with the same set of metric field equations. However, the latter scheme gets increasingly cumbersome with increasing complexity of Lagrangians at hand. Therefore, we shall use outline the Lagrange multiplier method appropriate to our investigation.

In summary, we will work out the Riemannian content of the generalized/modified gravitational Lagrangians which contain the scalars built out of the curvature tensor and its various contractions. Specifically, we will consider the gravitational Lagrangian densities of type $\mathcal{L}_G^{(n)} = f(\mathcal{S}) * 1$ where $\mathcal{S}$ is any scalar built out of curvature tensor and the function f is an arbitrary differentiable function of its argument. In that, we shall assume that n-form $\mathcal{S} * 1$ is

expressible in terms differential forms such as coframe and connection 1-forms, curvature 2-forms and their contraction using Hodge dual operator. As will be apparent, for practical convenience, we adopt the Lagrange multiplier method [21–26] using differential forms language (summarized in the appendix for convenience) adapted from [27] throughout the paper. Although the Lagrange multiplier method can also be carried out relative to a coordinate coframe, the use of differential forms enhances the practical use of the Lagrange multiplier method. Using the method that we adopted to derive metric field equations for $f(R)$ theory, Palatini $f(R)$ equations can be derived from the same set of variational derivatives, which can be considered as one of the advantages that makes the Lagrange multiplier is more efficient method. We shall work in $n \geqslant 3$ dimensions, and a gravitational action I_G has the general form

$$I_G = \int_U \mathcal{L}_G^{(n)} = \int_U f(\mathcal{S}) * 1 \tag{2}$$

in a local chart $U \subset M$ of n dimensional pseudo-Riemannian manifold where $\mathcal{L}_G^{(n)}$ is Lagrangian density n-form which already includes the invariant volume form and $\mathcal{S}$ is a scalar which can be expressed in terms of curvature tensor. The metric field equations that follow are chart independent provided that the basis coframe and the connection 1-forms have the following transformation properties

$$\theta^\alpha \mapsto \tilde{\theta}^\alpha = \Lambda^\alpha{}_\beta \theta^\beta, \tag{3}$$

$$\omega^\alpha{}_\beta \mapsto \tilde{\omega}^\alpha{}_\beta = (\Lambda^{-1})^\alpha{}_\mu \omega^\mu{}_\nu \Lambda^\nu{}_\beta + (\Lambda^{-1})^\alpha{}_\mu d\Lambda^\mu{}_\beta, \tag{4}$$

for overlapping charts, where the functions $\Lambda^\alpha{}_\gamma$ take the values in the Lorentz group. In our presentation, we shall suppress the integral sign and use Lagrangian density n-form for convenience. We start with deriving a general equation for the variational derivatives of generalized gravitational actions. The basic idea simply is to use product rule for the variational derivatives of the scalar fields and subsequently convert the variational derivatives of the scalars into variational derivatives of expressions involving forms which allows us to adopt orthonormal coframe. Relative to an orthonormal coframe, the total variational derivative of the n−form $\mathcal{L}_G^{(n)}$ can be calculated as follows. The Leibnitz rule for variational derivatives of the product of two scalars gives

$$\delta(f * 1) = (\delta f) * 1 + f \delta * 1, \tag{5}$$

where δ denotes variational derivative. The variational derivative of the function f can be written as $\delta f = f' \delta S$ where $f' = \frac{df}{d\mathcal{S}}$. Consequently, the variation can be written as

$$\delta \mathcal{L}_G^{(n)} = f'(\delta \mathcal{S}) * 1 + f \delta * 1. \tag{6}$$

Now, using the product rule for the variation

$$\delta(\mathcal{S} * 1) = (\delta \mathcal{S}) * 1 + \mathcal{S} \delta * 1 \tag{7}$$

for the first term on the right hand side of Eqn. (7), Eqn. (6) can be written as

$$\delta \mathcal{L}_G^{(n)} = f' \delta(\mathcal{S} * 1) + (f - f'\mathcal{S}) \delta * 1. \tag{8}$$

Note here that the second term on the right hand side of Eqn. (8) involves only the coframe variational terms whereas the variational derivatives of the gravitational variables, such as the connection 1-forms, come from the first term. The second term on the right hand side of Eqn. (8) can be simplified by recalling that relative to an orthonormal coframes θ^α, for $\alpha = 0, 1, 2 \cdots (n-1)$, the volume n-form can be written as

$$*1 = \tfrac{1}{n!} \varepsilon_{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n} \theta^{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n} \tag{9}$$

where $\varepsilon_{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n}$ completely antisymmetric permutation symbol. Using the expression (9), the variational derivative of the volume form takes the form

$$\delta * 1 = \tfrac{1}{n!} \varepsilon_{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n} \left(\delta\theta^{\alpha_1} \wedge \theta^{\alpha_2 \alpha_3 \cdots \alpha_n} + \theta^{\alpha_1} \wedge \delta\theta^{\alpha_2} \wedge \theta^{\alpha_3 \cdots \alpha_n} + \cdots + \theta^{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_{(n-1)}} \wedge \delta\theta^{\alpha_n} \right)$$
$$= \delta\theta^{\alpha_1} \wedge \tfrac{1}{(n-1)!} \varepsilon_{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n} \theta^{\alpha_2 \alpha_3 \cdots \alpha_n} \tag{10}$$

Note that, unlike exterior derivative, the variational derivative δ does not pick up a minus sign passing through a coframe 1-form and that δ does not commute with $*$. Thus, relabeling the dummy indices α_1 in (10) and the definition

of Hodge dualilty for basis coframe 1-forms, the expression for the coframe variation of the volume element in Eqn. (10) can be reduced to

$$\delta * 1 = \delta\theta^\alpha \wedge *\theta_\alpha. \tag{11}$$

Finally, using Eqn. (11) and Eqn. (8), the expression for the total variation of the action density $\mathcal{L}_G^{(n)}$ takes the following convenient form

$$\delta\mathcal{L}_G^{(n)} = f'\delta(\mathcal{S} * 1) + \delta\theta^\alpha \wedge (f - f'\mathcal{S}) * \theta_\alpha. \tag{12}$$

This key equation can be regarded as a kind of master formula which has a wide generality, as will be shown below. It expresses the variational derivative of the expression $f(\mathcal{S}) * 1$ in terms of the simpler variational derivative n-form $\mathcal{S} * 1$ plus additional terms. For instance, for the simplest case $\mathcal{S} = R$, which will be discussed in the next section, in which the Lagrangian density becomes $\mathcal{L}_{GEH}^{(n)} = f(R) * 1$ and the total variation will given by Eqn. (20), is the simplest example of the general expression (12). It is possible to consider slightly more general Lagrangian density where f is a function of a number scalars, say for example, $\mathcal{L}_G^{(n)} = f(\mathcal{S}, \mathcal{R}) * 1$. In this case, the total variational derivative takes the form

$$\delta\mathcal{L}_G^{(n)} = \frac{\partial f}{\partial \mathcal{S}}\delta(\mathcal{S} * 1) + \frac{\partial f}{\partial \mathcal{R}}\delta(\mathcal{R} * 1) - \delta\theta^\alpha \wedge \left(\frac{\partial f}{\partial \mathcal{S}}\mathcal{S} + \frac{\partial f}{\partial \mathcal{R}}\mathcal{R} - f\right) * \theta_\alpha, \tag{13}$$

where f is now assumed to be an analytical function of two distinct curvature scalars $\mathcal{S}$ and $\mathcal{R}$. Not to burden the presentation with unnecessary complications, we shall mainly consider the gravitational Lagrangian of type $f(\mathcal{S}) * 1$. In order to evaluate the total variational derivative of (12) with respect to independent connection and coframe 1-forms relative to an orthonormal basis explicitly, the first term on the right hand side of (12) is to be expressed in terms of coframe 1-forms, connection 1-forms, curvature 2-form and contractions of curvature 2-forms. The second term on the right hand side of (12) directly contributes to the coframe variational derivative and hence to the metric field equations. In the Lagrange multiplier method we shall use, the independent field variables are the coframe and connection 1-forms as well as the Lagrange multiplier $(n - 2)$-forms. However, the connection 1-forms are constrained to be torsion free using a Lagrange multiplier $(n - 2)$ form in n-dimensional space. The vanishing of the the torsion cannot be imposed straightforwardly to the variation derivatives since it is a dynamical constraint. In such a constrained theory, the coframe variations get contributions involving Lagrange multiplier forms.

The equivalence of the Lagrange multiplier method with the metric variation relative to a coordinate coframe has been shown in [28]. However, it should be emphasized that the Palatini method and the metric method, except for the only the case of the Einstein-Hilbert action $\mathcal{L}_{EH}^{(n)}$, lead to different field equations.

As remarked before, relative to an orthonormal coframe metric equations are induced by coframe variations and the equations obtained by independent connection variation can be used to solve the Lagrange multiplier $(n - 2)$-forms in terms of other fields. The Lagrange multiplier n-form terms, which contains vector-valued Lagrange multiplier $(n - 2)$-forms λ^α, of the form

$$\mathcal{L}_{LM}[\lambda_\alpha, \omega^\alpha{}_\beta, \theta^\alpha] = \lambda_\alpha \wedge \Theta^\alpha = \lambda_\alpha \wedge (d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta), \tag{14}$$

and λ_α is promoted to a new field variable. The variation of the total action is $\mathcal{L}_{tot}^{(n)} = \mathcal{L}_{GEH}^{(n)} + \mathcal{L}_{LM}$ with respect to the independent variables θ^α, $\omega_{\alpha\beta}$ and λ_α. The metric field equations of the original Lagrangian corresponds to the metric field equations of a section of the $\mathcal{L}_{tot}$ determined by $\Theta^\alpha = 0$ and the field equations obtained from connection variation allows one to express λ_α in terms of the other fields. The total variation of the total Lagrangian density $\mathcal{L}_{tot}^{(n)}$ with respect to the independent variables with the torsion constrained to vanish has the general expression

$$\delta\mathcal{L}_{tot}^{(n)} = \delta\theta^\alpha \wedge \left(\frac{\delta\mathcal{L}_G^{(n)}}{\delta\theta^\alpha} + D\lambda_\alpha\right) + \delta\omega_{\alpha\beta} \wedge \left\{\Pi^{\alpha\beta} - \tfrac{1}{2}(\theta^\alpha \wedge \lambda^\beta - \theta^\beta \wedge \lambda^\alpha)\right\} + \delta\lambda_\alpha \wedge \Theta^\alpha, \tag{15}$$

up to an omitted total exterior derivative and the antisymmetric $(n - 1)$-form $\Pi^{\alpha\beta} \equiv \delta\mathcal{L}_G^{(n)}/\delta\omega_{\alpha\beta}$ is defined for convenience. The metricity condition for the torsion free connection relative to an orthonormal coframe can be expressed in terms of connection 1-forms as $\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$. Therefore, the metricity constraint can be implemented by anti-symmetrizing the coefficient of $\delta\omega_{\alpha\beta}$ in expression for the total variation. However, a nonmetricity constraint for connection can also be imposed by introducing another Lagrange multiplier $(n - 1)$-form $\rho_{\alpha\beta} = \rho_{\beta\alpha}$ with the term $\rho^{\alpha\beta} \wedge (\omega_{\alpha\beta} + \omega_{\beta\alpha}) = \rho^{\alpha\beta} \wedge D\eta_{\alpha\beta}$ added to the original Lagrangian density. The Lagrange multipliers $\rho_{\alpha\beta}$ can be found from the symmetric part of the variational derivative with respect to the connection 1-form in this case. We will not

investigate the modified theories of gravity with torsion and non-metricity (metric-affine gauge theories of gravity) [22]. In this work, we simply assume that the connection is metric compatible and therefore it has the anti-symmetry property, $\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$ and accordingly $\delta\omega_{\alpha\beta} + \delta\omega_{\beta\alpha} = 0$ in all the total variations we consider. Consequently, one ends up with the property $\Pi_{\alpha\beta} + \Pi_{\beta\alpha} = 0$. There is an equivalence between the antisymmetric $\Pi_{\alpha\beta}$ which are tensor-valued $(n-1)$-forms and the Lagrange multiplier λ^α which are vector-valued $(n-1)$-forms [22]. The field equations $\delta\mathcal{L}^{(n)}_{tot}/\delta\omega_{\alpha\beta} = 0$ can uniquely be solved for the Lagrange multiplier $(n-2)$ forms as

$$\lambda^\alpha = 2i_\beta\Pi^{\beta\alpha} + \tfrac{1}{2}\theta^\alpha \wedge i_\mu i_\nu \Pi^{\mu\nu}, \tag{16}$$

by calculating two successive contractions of the equations $\delta\mathcal{L}^{(n)}_{tot}/\delta\omega_{\alpha\beta} = 0$. For $n = 3$, the wedge product in the second term (16) is unnecessary since λ^α is 1-form in three dimensions. The resulting expression then can be used in the $D\lambda^\alpha$ term for the metric equation $\delta\mathcal{L}^{(n)}_{tot}/\delta\theta_\alpha = 0$. The trace of this term, subject to the constraint $\Theta^\alpha = 0$, has the following general expression

$$\theta_\alpha \wedge D\lambda^\alpha = 2di_\beta(\theta_\alpha \wedge \Pi^{\alpha\beta}), \tag{17}$$

which applies to all the cases we consider. The equations $\delta\mathcal{L}^{(n)}_{tot}/\delta\lambda_\mu = \Theta^\mu = D\theta^\mu = 0$ imply that the resulting covariant exterior derivative D is torsion-free and metric compatible, and therefore $D^2\theta^\alpha = \Omega^\alpha{}_\beta \wedge \theta^\beta = 0$.

In the following sections, we shall explicitly work out the Riemannian content of the field equations obtained from metric variational principle for various generalized/modified gravitational actions and we shall assume that zero-torsion and metricity conditions are imposed for all the models. In particular, we make use of the general expressions (12) and (15)-(17). On the other hand we shall indicate how the field equations can be obtained adopting the Palatini approach in comparison. If a matter field couples not only to metric components [29] but also to the connection, the resulting theory is known as Riemann-Cartan theory where torsion is produced by spin density of the matter [25]. However, as has been noted before, torsion may also be generated by gradient of scalar field of a scalar-tensor theory, such as BD theory in which BD scalar field couples non-minimally to the curvature. On the basis of equivalence between $f(R)$ theory and the BD type scalar-tensor theory one also expects this to be the case for generic Palatini $f(R)$ theories. Although we shall not address the problem of formulating Riemann-Cartan type theory for the gravitational Lagrangians coupled with spinor matter Lagrangians, our unified approach is even appropriate to accommodate such an extension.

III. SOME PREVIOUS RESULTS: $f(R)$ THEORY

As an illustration of the general method of calculation for variational derivative of Lagrangian densities that we shall present, we first derive the field equations for the well-known $f(R)$ theory. In this section, we shall study both metric and Palatini $f(R)$ theories in relation to respective scalar-tensor theories.

A. Metric $f(R)$ Theory

We start by recalling that, relative to an orthonormal coframe, the Einstein-Hilbert n-form Lagrangian density can be written in the following equivalent forms

$$\mathcal{L}^{(n)}_{EH} = \tfrac{1}{2}\Omega_{\alpha\beta} \wedge *\theta^{\alpha\beta} = \tfrac{1}{2}R * 1, \tag{18}$$

where $*1 = \theta^{01\cdots(n-1)}$ is the oriented volume element, $R = i_\nu i_\mu \Omega^{\mu\nu} = i_\nu R^\nu$ is the scalar curvature and the dimension of the spacetime is assumed to have the values $n \geqslant 3$. From purely mathematical point of view, the Lagrangian density (18) can be assumed to have the generalization of the form

$$\mathcal{L}^{(n)}_{GEH} = f(R) * 1, \tag{19}$$

where the function f is assumed to be an arbitrary algebraic scalar function that is differentiable with respect to its argument. Similar to the formulae given in (12), the variational derivative of the total Lagrangian density $\delta\mathcal{L}^{(n)}_{GEH}$ can be written as

$$\delta\mathcal{L}^{(n)}_{GEH} = f'\delta\mathcal{L}^{(n)}_{EH} - \delta\theta^\alpha \wedge (f'R - f) * \theta_\alpha, \tag{20}$$

where $f' = \frac{df}{dR}$. The total variational derivative of the total Lagrangian density $\mathcal{L}_{tot}[\theta^\alpha, \omega^\alpha{}_\beta, \lambda_\alpha] = \mathcal{L}_{GEH}^{(n)} + \mathcal{L}_{LM}$ with respect to the independent variables can be written as

$$\delta\mathcal{L}_{tot}^{(n)} = \delta\omega_{\alpha\beta} \wedge \tfrac{1}{2}\left\{D(f' * \theta^{\alpha\beta}) - (\theta^\alpha \wedge \lambda^\beta - \theta^\beta \wedge \lambda^\alpha)\right\}$$
$$+ \delta\theta_\mu \wedge \left\{f'\Omega_{\alpha\beta} \wedge *\theta^{\alpha\beta\mu} + (f - Rf') * \theta^\mu + D\lambda^\mu\right\} + \delta\lambda_\alpha \wedge \Theta^\alpha, \tag{21}$$

up to a disregarded closed form, and from this expression one can read off

$$\Pi^{\alpha\beta} = D(f' * \theta^{\alpha\beta}). \tag{22}$$

Using the general formula (16) and the result (22), the Lagrange multiplier can be found as

$$\lambda^\alpha = 2 * (df' \wedge \theta^\alpha). \tag{23}$$

Consequently, using this expression for the Lagrange multiplier in the equation obtained from the coframe variations, one finds the metric field equations induced by the coframe variations as

$$- f' * G^\alpha + \tfrac{1}{2}(f - Rf') \wedge *\theta^\alpha + D * (df' \wedge \theta^\alpha) = 0, \tag{24}$$

which was actually derived long ago [14]. Here $*G^\alpha = -\tfrac{1}{2}\Omega_{\mu\nu} \wedge *\theta^{\alpha\mu\nu}$ is Einstein $(n-1)-$ form. An equation for the scalar function $f(R)$ can be found by tracing the metric equation (24). This gives

$$(n-1)d * df' = \left(\frac{n}{2}f - f'R\right) * 1, \tag{25}$$

which is identically satisfied for $f(R) = R$ for the vacuum case. Moreover, for the Einstein-Hilbert case where $f(R) = R$, the Lagrange multiplier vanishes identically since $\lambda^\alpha = 0$. For this case, the Palatini formalism leads to the same metric field equations as in the metric formalism (see Eqns. (27) and (28)) as well. We also note the special case $f(R) = R^2$ for which the trace equation (25) reduces to the homogeneous wave equation for scalar curvature $d * dR = 0$ in four spacetime dimensions $n = 4$. However, $R^2 * 1$ term survive for $n = 3$ and for $n > 4$. As can be observed from the field equations presented below, the spacetime dimension is usually a crucial parameter that may determine physical features of the model at hand. Note that, as expected on consistency grounds, for $f(R) = R$, Eqns. (24) yield the Einstein vacuum field equations. However, unlike Einstein field equations, they contain 4th order partial derivatives of the metric components relative to the coordinate coframe because of the presence of the term $D * (df' \wedge \theta^\mu)$. This term drops in the constant curvature case $\Omega^{\mu\nu} = k\theta^{\mu\nu}$. In this case, scalar curvature is constant, $R = kn(n-1)$ and $df' = 0$ identically. Thus, it is possible to find a Lagrangian density $f(R)$ for which isotropic spaces are vacuum solutions. Inserting $\Omega^{\mu\nu} = k\theta^{\mu\nu}$ into (24), one finds $f(R) = e^{k(n-1)R}$. Cosmological dynamics based on the exponential gravitational Lagrangian is studied in [30] and in generic $f(R)$ models $R = 0$ does not obviously imply $R = 0$ as in GR. In particular, we note here that, for another simple case $f(R) = R^m$ which we shall refer to in the succeeding sections, Eqn. (24) yields

$$- R * G^\alpha - \frac{(m-1)}{2m}R^2 * \theta^\alpha + \tfrac{1}{2}(m-1)(m-2)dR \wedge i^\alpha * dR + \tfrac{1}{2}(m-1)D * (dR \wedge \theta^\alpha) = 0, \tag{26}$$

which holds for $n \geqslant 3$ spacetime dimensions. The solution to the Einstein vacuum field equations $R^{\alpha\beta} = 0$ trivially solves the field equations derived from Lagrangian density $f(R) = R^m$. Thus, for example, static spherically symmetric solutions to the Einstein vacuum field equations are also solutions to (26). But the converse is not necessarily true. Not all the vacuum solutions to $f(R)$ theory are also solutions to the vacuum Einstein field equations.

B. Palatini/Riemann-Cartan Type $f(R)$ Theory

Riemann-Cartan type gravitational theories accommodate torsion, which is assumed to be generated by spin [31]. However, in addition to spin, scalar-tensor theories also give rise to torsion in terms of gradient of a scalar field [32]. As we shall show below, for generic $f(R)$ theory, the theory admits an algebraic torsion and consequently the Riemann-Cartan connection can be expressed in terms of Riemannian connection and a contorsion 1-form. In this case, the metric field equations can be cast into a form completely expressed in terms of Riemannian geometrical quantities. In the Palatini method to calculate the variational derivative to $f(R)$ theory, the field equations for connection is used to solve the connection in terms of Riemannian connection and additional terms resulting from algebraic torsion in terms of the gradient of the scalar df'. The Palatini approach to $f(R)$ theory, which we shall study in this section, is

therefore can be considered as modified gravitational theory with algebraic torsion. We shall start with formally the same Lagrangian density (18). In the case where non-zero torsion is allowed [33, 35], the field equations, which can be found from the variational derivative (21), are

$$df' \wedge *\theta^{\alpha\beta} + f'\Theta^{\mu} \wedge *\theta^{\alpha\beta}{}_{\mu} = 0, \tag{27}$$

$$f' * G^{\alpha} + (f - f'R) * \theta^{\alpha} = 0, \tag{28}$$

respectively. Note here that for $f(R) = R$, the Palatini variation yields $\Pi^{\alpha\beta} = 0$ identically if one imposes $\Theta^{\alpha} = 0$ after the variation in the case of $\mathcal{L}_{EH}^{(n)}$. This follows from the fact that in the metric variation for the Einstein-Hilbert Lagrangian, one finds from (23) that $\lambda^{\alpha} = 0$ [36]. As a result Palatini and metric variational derivatives lead to the identical metric equations in this exceptional case of the Einstein-Hilbert Lagrangian. The field equation which follows from the independent connection variation, Eqn. (27) is an algebraic equation for the torsion 2-form Θ^{α} which can be solved for it as

$$\Theta^{\alpha} = \frac{1}{(n-2)}\theta^{\alpha} \wedge d\ln f', \tag{29}$$

using the identity $\theta^{\alpha} \wedge i_{\alpha}\lambda = p\lambda$ which holds for arbitrary p-form λ. This result then can be used to solve the corresponding connection 1-forms which can be obtained from the Cartan's first structure equations.

Returning to the Palatini equations, the variational derivative with respect to connection, namely $\Pi^{\alpha\beta} = 0$ lead to an algebraic torsion in terms of the gradient df'. As a result, in Palatini formalism, even if the connection is metric compatible (and therefore commutes with contractions), it has torsion. However, algebraic torsion allows the elimination of the independent connection 1-form and the field equations can be cast into a metric theory with additional gravitational currents under suitable conditions stated below. For this purpose, it is convenient to introduce contorsion 1-form $K_{\alpha\beta} = -K_{\beta\alpha}$ defined by $\Theta^{\alpha} = K^{\alpha}{}_{\beta} \wedge \theta^{\beta}$. The contorsion 1-forms can be expressed in term of the exterior derivatives of the function f' as

$$K^{\alpha\beta} = \frac{1}{(n-2)}\left\{\theta^{\alpha}(d\ln f')^{\beta} - \theta^{\beta}(d\ln f')^{\alpha}\right\}, \tag{30}$$

where the indices of the 1-form $d\ln f'$ are relative to the orthonormal coframe we employ. In terms of contorsion 1-form, the metric compatible connection 1-form $\dot{\omega}^{\alpha}{}_{\beta}$ can be written as sum of Riemannian connection $\omega^{\alpha}{}_{\beta}$ and a contorsion term as

$$\omega^{\alpha}{}_{\beta} = \dot{\omega}^{\alpha}{}_{\beta} + K^{\alpha}{}_{\beta}, \tag{31}$$

where, in order to distinguish metric connection from the Riemannian connection, the dotted symbols for the Riemannian quantities are used. The result (31) will be elaborated below.

The Cartan's second structure equations for the corresponding curvature 2-form then lead to similar decomposition of the curvature 2-forms as well. One finds

$$\Omega^{\alpha\beta} = \dot{\Omega}^{\alpha\beta} + \dot{D}K^{\alpha\beta} + K^{\alpha}{}_{\mu} \wedge K^{\mu\beta}, \tag{32}$$

where $\dot{\Omega}^{\alpha\beta}$ is the Riemannian curvature 2-form and $\dot{D}$ is the covariant exterior derivative corresponding to Riemannian covariant derivative. Note that f and hence f' is a function of scalar curvature R. Thus, the elimination of independent connection requires R to be expressed in terms of the dotted quantities using the trace of the metric equations (28). If we assume that this requirement can be satisfied, then the Einstein $(n-1)$-form can be written as

$$f' * G^{\alpha} = f' * \dot{G}^{\alpha} - \dot{D} * (df' \wedge \theta^{\alpha}) - \frac{(n-1)}{(n-2)}\frac{1}{f'} * T^{\alpha}[f'], \tag{33}$$

where the energy momentum forms for the scalar field f', namely

$$* T_{\alpha}[f'] = -\frac{1}{2}\left\{(i_{\alpha}df') * df' + df' \wedge i_{\alpha} * df'\right\}, \tag{34}$$

is defined for convenience. Note that this result explicitly shows that non-zero torsion produces the kinetic term for the function f' at the level of action and this corresponds to the effective energy momentum term $*T^{\mu}[f']$ in the resulting metric equation. Finally, using this result and dropping the dots over Riemannian quantities, the metric field equations can be written as

$$f' * G^{\alpha} = \frac{1}{2}(f - Rf') * \theta^{\alpha} + D * (df' \wedge \theta^{\alpha}) + \frac{(n-1)}{(n-2)}\frac{1}{f'} * T^{\alpha}[f'], \tag{35}$$

which are expressed in terms of Riemannian connection 1-form and the corresponding Riemannian curvature tensor. Taking the trace of the metric field equation one finds

$$(n-1)d*df' = \tfrac{1}{2}(n-1)\frac{1}{f'}df' \wedge *df' + \tfrac{1}{2}f'R*1 \tag{36}$$

Note that because of the term $D*(df' \wedge \theta^\alpha)$, the field equations (35) are fourth order partial differential equations relative to a coordinate basis. The above reduction of $f(R)$ theory with torsion to a metric theory has been carried out recently using the equivalence of such a theory with Scalar-Tensor theories relative to a coordinate basis and it is also argued that the theories with torsion and/or non-metricity can also be cast into metric theory for $f(R)$ theories [33]. As we shall show explicitly in the following sections, this is not true for other modified gravitational Lagrangians.

We now briefly discuss the equivalence of generic $f(R)$ theory with torsion to the Scalar-tensor theory. Note that if the field variable $f'(R)$ is promoted to a scalar field, then equations of motion (24) and (25) for $f(R)$ theory have formal resemblance to the metric equations in BD theory except for energy momentum $(n-1)$-form corresponding to the kinetic term for the BD scalar function. In other words, the equations of motion for generic $f(R)$ theory can be obtained from a Brans-Dicke type action with kinetic term for BD scalar that minimally couples to the metric tensor is absent. Consider the following BD-type Lagrangian density in n dimensions

$$\mathcal{L} = \mathcal{L}_{BD} + V(\phi)*1 = -\frac{\phi}{2}\Omega_{\mu\nu} \wedge *\theta^{\mu\nu} + \frac{\omega}{2\phi}d\phi \wedge *d\phi + V(\phi)*1, \tag{37}$$

where the scalar field ϕ couples to the metric non-minimally and there is a self interaction potential term for the scalar field. The metric field equations, that follow from (37), is

$$\phi*G^\alpha = -\frac{\omega}{\phi}*T^\alpha[\phi] + D*(d\phi \wedge \theta^\alpha) + V(\phi)*\theta^\alpha, \tag{38}$$

where the canonical energy momentum $(n-1)$-form field $*T^\alpha[\phi]$, which results from commuting the coframe variation with Hodge dual, is

$$*T_\alpha[\phi] = -\tfrac{1}{2}\{(i_\alpha d\phi)*d\phi + d\phi \wedge i_\alpha *d\phi\} \tag{39}$$

whereas the field equation for scalar field can be written as

$$\left(\omega + \tfrac{n-1}{n-2}\right)d*d\phi = V'*1. \tag{40}$$

For the BD theory with vanishing BD-parameter, namely for $\omega = 0$, the metric and the scalar field equations that follow from the variation of the action for this Lagrangian with zero torsion condition imposed can be written as

$$-\phi*G^\alpha + D*(d\phi \wedge \theta^\alpha) + V(\phi)*\theta^\alpha = 0, \tag{41}$$
$$R = V'(\phi), \tag{42}$$

respectively [34]. Here, $'$ denotes derivative with respect to the argument of the scalar potential, $V' = \frac{dV}{d\phi}$. The trace of the metric equation (41) and (42) can be combined to have

$$(n-1)d*d\phi = \left\{nV(\phi) - \tfrac{1}{2}(n-2)\phi V'(\phi)\right\}*1. \tag{43}$$

Comparison with the metric field equations for the $f(R)$ theory, assuming that f is a definite functional form and that $f'' \neq 0$, the equation $f'(R) = \phi$ can be inverted in order to be express R in terms of ϕ. Then, $R(\phi)$ can be used in the equation

$$nV(\phi) - \tfrac{1}{2}(n-2)\phi V'(\phi) = \tfrac{n}{2}f - f'R, \tag{44}$$

to determine the scalar potential $V(\phi)$. In particular, for $n = 4$ one has $V(\phi) = \tfrac{1}{2}f(R(\phi))$. As a result, with the above identification and redefinition of the field variables, the field equations for $f(R)$ theory transformed into a BD-type theory. In the BD-type scalar theory with field Eqns. (41) and (42), the energy momentum $(n-1)$-forms for any matter field $*T_m^\alpha[\Phi]$ that couples only to the metric field but not to the derivatives of the metric field, are covariantly constant. Explicitly, using the contracted second Bianchi identity, which is a purely geometrical identity, namely $D*G^\alpha = 0$, the covariant derivative of (41) gives

$$-d\phi \wedge *G^\alpha + \Omega^\alpha{}_\beta \wedge *(d\phi \wedge \theta^\beta) + V'(\phi)d\phi \wedge *\theta^\alpha = 0, \tag{45}$$

by virtue of the field equation (42). In other words, $D * T_m^\alpha[\Phi] = 0$ follows from the fact that the covariant exterior derivative of the vacuum metric field equations (41) vanish. Conversely, the metric field equations (41) yield the field equation (43) by taking the covariant exterior derivative of (41) and using (42). The fact that the field equations are covariantly constant can be shown to follow from more general property of Lagrangian density that it has diffeomorphism invariance, independent of the resulting field equations.

It is important to emphasize again that, the Palatini and the metric variational principles in general, lead to different sets of metric field equations. The scalar tensor equivalence for the corresponding $f(R)$ theories above provides an example for the inequivalence of the two theories since the Palatini variation $f(R)$ theory lead BD-type scalar tensor theory with BD parameter $\omega = -\frac{(n-1)}{(n-2)}$ with potential term for the scalar field whereas $f(R)$ theory based on metric variational principle lead to scalar-tensor theory with $\omega = 0$ [33, 40].

The Palatini connection is conformally related to Levi-Civita as in (31) with the contorsion form expressed in terms of a scalar gradient (30). Because the conformal factor involves scalar curvature, the parallel transport of the connection in the Palatini method leads to a distinct autoparallel curves which can be expressed in terms of Levi-Civita connection. A timelike vector field X, which is normalized as $g(X, X) = -1$, is tangent to autoparallel curves of the connection (31) if it satisfies

$$\nabla_X X = 0. \tag{46}$$

Here, it is convenient to introduce the operator $\flat$ which maps the vectors to their metric duals, i.e. to the corresponding 1-forms. It can be defined implicitly in terms of the metric tensor as $X^\flat(-) = g(X, -)$. Since ∇_X is metric compatible and hence commutes with the operator $\flat$, we have $\nabla_X X^\flat = (\nabla_X X)^\flat$. Using the expression (31) for the connection 1-form, Eqn. (46) can be written in terms of the Riemannian covariant derivative $\dot\nabla_X$ as

$$\dot\nabla_X X^\flat + \frac{f''}{2f'} i_X(dR \wedge X^\flat) = 0. \tag{47}$$

As a result, similar to the scalar tensor theories [37], the non-Riemannian connection given in (31) induces an acceleration term depending on the gradient of the scalar curvature into the autoparallel curves. We note here that the geodesics are determined by the Riemannian structure instead of the independent Palatini connection as a result of the fact that the matter energy momentum $(n-1)$-forms are covariantly constant: $D * T_m^\alpha = 0$ [38]. On the other hand, the correct cosmological behavior of $f(R)$ theories in the Palatini formalism requires consistent averaging procedure of energy momentum forms describing the microscopic(atomic) structure of matter fields and it also predicts alterations to atomic structure and particle physics laws, see e.g. [39].

C. Equivalence to Scalar-tensor Theories at the Level of Action

The equivalence of $f(R)$ theory to particular Scalar-tensor theories discussed above by comparing the corresponding field equations can also be deduced, perhaps in an indirect way, at the level of action. (Such an equivalence holds for $f(R)$ theory with/without torsion.) The equivalence can be achieved by introducing an auxiliary scalar field χ and a scalar Lagrange multiplier λ imposing the constraint $\chi - \mathcal{S} = 0$ to the gravitational action.

We shall mainly consider modified gravitational Lagrangians of the form $f(\mathcal{S}) * 1$, where f is an algebraic function of scalar $\mathcal{S}$ and we assume that $S * 1$ can be expressible in terms of basis coframe forms, connection 1-forms, curvature 2-forms and their various contractions. In the place of $f(\mathcal{S}) * 1$, consider an auxiliary Lagrangian density $\mathcal{L}_{aux} = \mathcal{L}_{aux}[\chi, \lambda, \theta^\alpha, \omega^\alpha_{\ \beta}]$ of the form

$$\mathcal{L}_{aux} = [f(\chi) + \lambda(\chi - \mathcal{S})] * 1. \tag{48}$$

The total variational derivative of the auxiliary Lagrangian (48) with respect to the independent variables indicated above yields

$$\delta\mathcal{L}_{aux} = [\delta\chi(f' + \lambda) + \delta\lambda(\chi - \mathcal{S})] * 1 - \lambda\delta(\mathcal{S} * 1) + \delta\theta^\alpha \wedge (f + \lambda\mathcal{S}) * \theta_\alpha \tag{49}$$

where δ denotes the variational derivative as before and the prime denotes derivative with respect to the auxiliary field χ. The equation for the scalar field χ immediately yield the Lagrange multiplier $\lambda = -f'$. Replacing this result and the corresponding constraint back into the action (48) one finds the dynamically equivalent Lagrangian density as

$$\mathcal{L}_{aux-eqv.} = [f(\chi) + (\chi - \mathcal{S})f'(\chi)] * 1, \tag{50}$$

up to an overall sign. The resulting Lagrangian density $\mathcal{L}_{aux-eqv.}$ now has no dependence on the Lagrange multiplier λ and in the equivalent Lagrangian density f' couples to the scalar $\mathcal{S}$. However, the reduced form of the auxiliary Lagrangian density allows redefinition of the field variables. For $f(R)$ theory, namely for $\mathcal{S} = R$, Eqns. (49) and (50) reduce to

$$\delta\mathcal{L}_{aux} = [\delta\chi(f' + \lambda) + \delta\lambda(\chi - R)] * 1 - \lambda\delta(R * 1) - \delta\omega_{\mu\nu} \wedge D(\lambda * \theta^{\mu\nu})$$
$$+ \delta\theta^{\alpha} \wedge \left\{ -\lambda\Omega_{\mu\nu} \wedge *\theta^{\mu\nu}{}_{\alpha} + (f + \lambda R) * \theta_{\alpha} \right\}, \tag{51}$$

and

$$\mathcal{L}_{aux-eqv.} = f'\Omega_{\alpha\beta} \wedge *\theta^{\alpha\beta} + (f - f'R) * 1, \tag{52}$$

respectively. Although the auxiliary action (52) consistently simplifies to the original action, this form of the action further suggests the redefinition $f'(R) = \phi$. This allows one to bring the action (52) into equivalent Scalar-tensor type action given in (37) with $\omega = 0$ and with an appropriate potential term for the scalar field assuming that ϕ can be written in terms of R. This general mechanism of generating non-minimally coupled scalar field in generalizing gravitational actions will be used to propose a modified Chern-Simons theory in four dimensions in a later section.

IV. VARIATIONAL DERIVATIVES OF MODIFIED GRAVITATIONAL ACTIONS

In addition to the generic $f(R)$ theories, modified gravitational theories based on the Lagrangian densities involving quadratic curvature invariants and their algebraic generalizations also lead to fourth order models. These models have a long history [41], they have been studied in different contemporary contexts ranging from cosmology and perturbative approach to quantum theory gravity [42]. This section is devoted to the calculation of the metric field equations for fourth and sixth order theories.

A. Generalized Gauss-Bonnet Actions

The outcome of employing extra dimensions for unification as well as quantum corrections of gravity requires some well behaving generalizations of the EH Lagrangian, such as scalars constructed from of higher order curvature terms. Actually, using an arbitrary scalar in action constructed from Riemann tensor leads to pathologies [43, 44] such as to have more than required degrees of freedom in equations of motion or existence of ghosts. However, there is a special combination of such terms in action, as shown by Lovelock [45], which are known as dimensionally continued Euler forms or Lovelock scalars. It turns out that, the Lagrangians constructed from these terms have some desirable properties such as, though nonlinearly, the field equations involve at most second derivatives of metric components [45, 46], theory is free of ghosts [43, 48], i.e. kinetic terms in the action with the wrong sign, and in four dimensions the only surviving term apart from cosmological constant is the EH Lagrangian [45, 49]. This generalization of considering Euler forms were studied in different contexts [50–55] such as Kaluza-Klein reduction, low energy limit of string theory, brane-worlds, and higher dimensional exact solutions.

Thus, in higher dimensions, one can consider, in addition to the usual Einstein-Hilbert term, the combination of Euler forms

$$I_G = \sum_{k=0}^{[n/2]} \mathcal{L}_L^{(k,n)}, \tag{53}$$

where

$$\mathcal{L}_L^{(k,n)} = \Omega_{\alpha_1\beta_1} \wedge \cdots \wedge \Omega_{\alpha_k\beta_k} \wedge *\theta^{\alpha_1\beta_1\cdots\alpha_k\beta_k}. \tag{54}$$

For $k = 2n$, $\mathcal{L}_L^{(n,n)}$ is an exact form and is proportional to Euler-Poincare characteristic of even dimensional manifold. For simplicity, we consider the gravitational Lagrangian density n-form

$$\mathcal{L}_{\mathcal{G}}^{(n)} = \Omega_{\alpha\beta} \wedge \Omega_{\mu\nu} \wedge *\theta^{\alpha\beta\mu\nu}$$
$$= 2\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} - 4R_{\alpha} \wedge *R^{\alpha} + R^2 * 1, \tag{55}$$

corresponding to $k = 2$ for $n \geqslant 4$. This term is called as Gauss-Bonnet or Lanczos Lagrangian, it is a pure divergence in $d = 4$, and first introduced by Lanczos [46]. Although here we only consider Gauss-Bonnet term, it is straightforward

to extend the formalism to the higher order Euler form Lagrangian densities. Now, it is convenient to define the scalar $\mathcal{G} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ which separates the volume form so that $\mathcal{L}_{\mathcal{G}}^{(n)} = \mathcal{G} * 1$. In four dimensions, $\mathcal{L}_{\mathcal{G}}^{(4)}$ is a total differential and does not contribute to the field equations. This follows from the identity

$$\tfrac{1}{2}\varepsilon^{\alpha\beta}{}_{\mu\nu}\Omega^{\mu\nu} = \Omega^{\alpha\beta} - \tfrac{1}{2}(\theta^\alpha \wedge R^\beta - \theta^\beta \wedge R^\alpha) + \tfrac{1}{4}R\theta^{\alpha\beta}, \tag{56}$$

which relates the left dual of the curvature 2-form to the curvature 2-form itself and its contractions [47]. From (56), it follows that

$$\Omega_{\alpha\beta} \wedge \Omega_{\mu\nu} \wedge *\theta^{\alpha\beta\mu\nu} = -\tfrac{1}{2}\Omega^\alpha{}_\beta \wedge \Omega^\beta{}_\alpha, \tag{57}$$

as a result of first Bianchi identity and the right hand side is an exact form (see Eqn. (107)). Therefore, the GB term of the form $\Omega_{\alpha\beta} \wedge \Omega_{\mu\nu} \wedge *\theta^{\alpha\beta\mu\nu}$ does not contribute to the field equations. However, it is possible to make $\mathcal{L}_{\mathcal{G}}^{(4)}$ to contribute to field equations, for example, by coupling it with a scalar field. On the other hand, the presentation of $f(R)$ type Lagrangian densities indicates that $f(\mathcal{G}) * 1$ type Lagrangian density will also have similar features that f' will act like a non-minimally coupled scalar field. After these preliminary remarks, we will derive the equations of motion for the modified Lagrangian density of the form

$$\mathcal{L}_{G\mathcal{G}}^{(n)} = f(\mathcal{G}) * 1, \tag{58}$$

for $n \geqslant 4$. The independent variations with respect to the coframe and connection 1-forms are constrained to be torsion free by Lagrange multiplier term (14). Using the same steps in the derivation of the variation (12) from the modified Lagrangian (19), the total variation of the Lagrangian density,

$$\mathcal{L}_{tot}^{(n)} = \mathcal{L}_{G\mathcal{G}}^{(n)} + \mathcal{L}_{LM}, \tag{59}$$

is related to the variational derivative of the Lagrangian density $\mathcal{L}_{\mathcal{G}}^{(n)}$ as

$$\delta\mathcal{L}_{tot}^{(n)} = f'\delta\mathcal{L}_{\mathcal{G}}^{(n)} - \delta\theta^\alpha \wedge (f'\mathcal{G} - f) * \theta_\alpha + \delta\mathcal{L}_{LM}. \tag{60}$$

Note that for $n = 4$, there is no contribution from the term containing $\delta\mathcal{L}_{\mathcal{G}}^{(n)}$ on the RHS of (60). Thus, the cases $n > 4$ and $n = 4$ lead to different field equations. The variations on the right hand side of this expression can be calculated to find

$$\delta\mathcal{L}_{tot}^{(n)} = \delta\omega_{\alpha\beta} \wedge \left\{\Pi^{\alpha\beta} - \tfrac{1}{2}(\theta^\alpha \wedge \lambda^\beta - \theta^\beta \wedge \lambda^\alpha)\right\} + \delta\lambda_\alpha \wedge \Theta^\alpha$$
$$+ \delta\theta_\gamma \wedge \left\{f'\Omega_{\mu\nu} \wedge \Omega_{\alpha\beta} \wedge i^\gamma * \theta^{\mu\nu\alpha\beta} + (f'\mathcal{G} - f) * \theta^\gamma + D\lambda^\gamma\right\}. \tag{61}$$

Here $'$ denotes derivatives with respect to the scalar $\mathcal{G}$ and the expression for $\Pi_{\alpha\beta}$ obtained from the variational derivative with respect to the connection 1-form yields

$$\Pi^{\alpha\beta} = 2D\left(f'\Omega_{\mu\nu} \wedge *\theta^{\mu\nu\alpha\beta}\right), \tag{62}$$

which is to be evaluated subject to the vanishing torsion condition. In contrast to the other field equations that we shall consider below, the field equations for the GB terms are second order in metric components. This follows from the fact that the second Bianchi identity satisfied by the curvature 2-forms and the Lagrange multiplier terms can be expressed in terms of curvature 2-forms. The order of the metric equations persist even in the corresponding first order formalism where an independent dynamical torsion is present.

As a result, the expressions for the Lagrange multipliers can be inserted into the metric equations $\delta\mathcal{L}_{tot}^{(n)}/\delta\theta_\lambda = 0$ to find

$$f'\Omega_{\mu\nu} \wedge \Omega_{\alpha\beta} \wedge *\theta^{\mu\nu\alpha\beta\lambda} + (f - f'\mathcal{G}) * \theta^\lambda - 4\Omega_{\mu\nu} \wedge D * (df' \wedge \theta^{\mu\nu\lambda}) = 0, \tag{63}$$

for $n > 4$ dimensions. The field equations (63) that follow from the $f(\mathcal{G})$ Lagrangian are similar to the field equations for the Lagrangians with a scalar field which couple non-minimally to the GB-terms. The term $\Omega_{\mu\nu} \wedge D * (df' \wedge \theta^{\mu\nu\lambda})$ contains 4th order partial derivatives of the metric components relative to a coordinate coframe. Now specializing to $n = 4$ dimensions, in particular, to the gravitational Lagrangian $\mathcal{L}_{tot}^{(4)} = \mathcal{L}_{EH}^{(4)} + \mathcal{L}_{GGB}^{(4)}$, the metric field equations can be written as

$$- *G^\alpha + (f - f'\mathcal{G}) * \theta^\alpha - 4\Omega_{\mu\nu} \wedge D * (df' \wedge \theta^{\mu\nu\alpha}) = 0. \tag{64}$$

Apparently, the order of the metric field equations for $f(\mathcal{G})$ augmented to four as a result of algebraic generalization of GB term. We refer to the review article [17] for $f(\mathcal{G}$ models and their applications.

Before we begin to study the the individual terms appearing in (55), we note that the dimension of spacetime is crucial and in particular, in four spacetime dimensions, using (55) and (56) the most general Lagrangian which are quadratic in the scalars the curvature can be written in the form

$$\mathcal{L}_G^{(4)} = aR^\alpha \wedge *R_\alpha + bR^2 *1, \tag{65}$$

with a, b being constants. The Lagrangian density (65) is the most general gravitational Lagrangian density involving quadratic curvature terms in three spacetime dimensions as well and we will study of this special case in a later section below. However, the most general Lagrangian involving quadratic curvature terms is to include $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ term in $n > 4$ dimensions.

B. Ricci-Squared Action

In this subsection we will calculate the variational derivatives of the Ricci-squared Lagrangian density whereby the corresponding field equations for the gravitational Lagrangian densities in (65) with arbitrary a, b and its algebraic generalizations of the form $f(R, \mathcal{P})$ can be found using the results of the previous sections. We first consider a Lagrangian density containing Ricci tensor-squared term, which can be written in terms of Ricci 1-forms as follows

$$\mathcal{L}_{\mathcal{P}}^{(n)} = R_\alpha \wedge *R^\alpha = R_{\alpha\beta}R^{\alpha\beta} *1. \tag{66}$$

The metric field equations corresponding to $\mathcal{L}_{\mathcal{P}}^{(n)}$ will also be used later in the study of three dimensional massive gravity models. We first derive field equations for the Lagrangian density $\mathcal{L}_{\mathcal{P}}^{(n)}$ in a form appropriate to our presentation and then derive the field equations for its generalization of the form $\mathcal{L}_{G\mathcal{P}}^{(n)} = f(\mathcal{P}) *1$ where $\mathcal{P} \equiv R_{\alpha\beta}R^{\alpha\beta}$. We first note that the symmetry of the Ricci tensor indices $R^\alpha \wedge \theta_\alpha = 0$ follows from the contraction of the first Bianchi identity, namely $D\Theta^\alpha = \Omega^\alpha{}_\beta \wedge \theta^\beta = 0$ together with the anti-symmetry property $\Omega_{\alpha\beta} + \Omega_{\beta\alpha} = 0$ of the curvature 2-forms relative to an orthonormal coframe. Thus, the symmetry of the components of the Ricci tensor can be incorporated conveniently into the total variation without introducing an additional constraint. We constrain the variation to be torsion-free so that we have the total Lagrangian density $\mathcal{L}_{tot}^{(n)} = \mathcal{L}_{\mathcal{P}}^{(n)} + \mathcal{L}_{LM}$. This Lagrangian has the total variation

$$\begin{aligned}
\mathcal{L}_{tot}^{(n)} = {}& \delta\omega_{\alpha\beta} \wedge \left\{ D * (\theta^\alpha \wedge R^\beta - \theta^\beta \wedge R^\alpha) - \tfrac{1}{2}(\theta^\alpha \wedge \lambda^\beta - \theta^\beta \wedge \lambda^\alpha) \right\} + \delta\lambda_\alpha \wedge \Theta^\alpha \\
& + \delta\theta^\mu \wedge \left\{ \Omega_{\alpha\beta} \wedge i_\mu * (\theta^\alpha \wedge R^\beta - \theta^\beta \wedge R^\alpha) - i_\mu(R_\alpha \wedge *R^\alpha) + D\lambda_\mu \right\},
\end{aligned} \tag{67}$$

where the identity $R_{\mu\nu} *1 = \Omega_{\alpha\mu} \wedge *\theta^\alpha{}_\nu$ has been used to obtain (67). The resulting vacuum metric field equations, subject to the condition that $\Theta^\alpha = 0$, can be conveniently written in terms of 1-form $\mathcal{P}^\alpha = \mathcal{P}^\alpha{}_\beta \theta^\beta$ as $*\mathcal{P}^\alpha = 0$ where $*\mathcal{P}^\alpha$ is given by

$$*\mathcal{P}^\mu = \Omega_{\alpha\beta} \wedge i^\mu * (\theta^\alpha \wedge R^\beta - \theta^\beta \wedge R^\alpha) - i^\mu(R_\alpha \wedge *R^\alpha) + 2Di_\alpha D * (\theta^\alpha \wedge R^\mu - \theta^\mu \wedge R^\alpha). \tag{68}$$

To our knowledge, the metric field equation for the Ricci-squared model formulated in terms of differential forms (68) is new. The covariant exterior derivatives in the Lagrange multiplier terms explicitly show that the Ricci-squared model is a fourth order model. 1-form $\mathcal{P}_\mu$ defined by (68) is symmetric with respect to its indices $\mathcal{P}_{\alpha\beta} = \mathcal{P}_{\beta\alpha}$. We also note here that the trace of the metric field equations (68) can be found as

$$\mathcal{P}^\alpha{}_\alpha *1 = (n-4)R^\alpha \wedge *R_\alpha + \tfrac{n}{2}d * dR. \tag{69}$$

The trace formula (69) again reflects the fact that the spacetime dimension is important for the Ricci-squared models as we shall see in three dimensions. This form of the field equations allows one to find a nontrivial solution to the Λ-vacuum field equations almost by inspection of the metric field equations. For example, $R^\alpha = k\theta^\alpha$ with the constant k solves the $\Lambda-$vacuum field equations $*\mathcal{P}^\alpha = -2(n-1)k^2 * \theta^\alpha$.

Now, having the total variational derivative of $\mathcal{L}_{\mathcal{P}}^{(n)}$ at our disposal, the field equations for $\mathcal{L}_{G\mathcal{P}}^{(n)} = f(\mathcal{P}) * 1$ can be found. As before, we also assume that f is an algebraic and differentiable function of its argument. The corresponding metric field equations can be found by straightforward application of (12). The total variation of $\mathcal{L}_{tot}^{(n)} = \mathcal{L}_{G\mathcal{P}}^{(n)} + \delta\mathcal{L}_{LM}$ then becomes

$$\delta\mathcal{L}_{tot}^{(n)} = f'\delta\mathcal{L}_{\mathcal{P}}^{(n)} - \delta\theta^\alpha \wedge (f'\mathcal{P} - f) * \theta_\alpha + \delta\mathcal{L}_{LM}. \tag{70}$$

Carrying out the total variation using (12) one can find

$$\delta\mathcal{L}_{tot}^{(n)} = \delta\omega_{\alpha\beta} \wedge \left\{ D * f'(R^\alpha \wedge \theta^\beta - R^\beta \wedge \theta^\alpha) - \tfrac{1}{2}(\theta^\alpha \wedge \lambda^\beta - \theta^\beta \wedge \lambda^\alpha) \right\} + \delta\lambda_\alpha \wedge \Theta^\alpha$$
$$+ \delta\theta^\mu \wedge \left\{ 2f'\Omega_{\alpha\beta} \wedge i_\mu * (R^\alpha \wedge \theta^\beta) - f' i_\mu(R_\alpha \wedge *R^\alpha) + (f'\mathcal{P} - f) * \theta_\mu + D\lambda_\mu \right\}. \tag{71}$$

For the purpose of comparison, it is convenient to write the resulting metric field equation using 1-form $\mathcal{P}_\alpha$ defined in (68). This gives

$$f' * \mathcal{P}^\alpha + D(f'\lambda^\alpha) + (f'\mathcal{P} - f) * \theta^\alpha = 0. \tag{72}$$

In contrast to the metric formalism, if the Palatini formalism is used as in [18], then from (67), the equation for the connection can be found as

$$\Pi^{\alpha\beta} = D * f'(\theta^\alpha \wedge R^\beta - \theta^\alpha \wedge R^\beta) = 0, \tag{73}$$

where D in this case is non-Riemannian covariant exterior derivative. Eqn. (73) has to be solved for connection 1-forms and then subsequently replaced into the coframe variation to obtain the metric field equations. Obviously, as in the case of $f(R)$ theory, it follows from (73) that the Palatini and metric formalism lead to totally different set of equations for the metric field. However, adopting Palatini formalism with an independent and *torsion-free* connection, the recent work on the models based on the Lagrangian n-form of the generic form $f(\mathcal{P}) * 1$ lead to the result that such models accommodate metrics of Einstein spaces [56].

Returning to our treatment of the models with quadratic curvature invariants using the metric formalism, we now study another popular fourth order model with the Lagrangian density containing Riemann tensor-squared (Krectschmann scalar) terms [57], namely the n-form

$$\mathcal{L}_\mathcal{R}^{(n)} = \tfrac{1}{2}\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} = \tfrac{1}{4}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} * 1, \tag{74}$$

and its algebraic generalization is of the form

$$\mathcal{L}_{G\mathcal{K}}^{(n)} = f(\mathcal{K}) * 1, \tag{75}$$

with f being an algebraic function of the Krectschmann scalar $\mathcal{K} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. The field equations for this Lagrangian, with torsion tensor constrained to vanish, can be obtained from the variation of the total Lagrangian density

$$\mathcal{L}_{G\mathcal{K}}^{(n)} = \mathcal{L}_\mathcal{K}^{(n)} + \mathcal{L}_{LM}. \tag{76}$$

For the Lagrangian density $\mathcal{L}_{G\mathcal{R}}^{(n)}$, equation (12) becomes

$$\delta\mathcal{L}_{G\mathcal{K}}^{(n)} = f'\delta\mathcal{L}_\mathcal{K}^{(n)} - \delta\theta^\alpha \wedge \tfrac{1}{2}(f'\mathcal{K} - f) * \theta_\alpha + \delta\mathcal{L}_{LM}. \tag{77}$$

The total variation with respect to independent coframe and connection 1-forms and Lagrange multiplier $(n-2)$-forms then can be written as

$$\delta\mathcal{L}_{tot}^{(n)} = \delta\omega_{\alpha\beta} \wedge \left\{ D * (f'\Omega^{\alpha\beta}) + \tfrac{1}{2}(\lambda^\alpha \wedge \theta^\beta - \lambda^\beta \wedge \theta^\alpha) \right\} + \delta\lambda_\alpha \wedge \Theta^\alpha$$
$$+ \delta\theta^\mu \wedge \left\{ -\tfrac{1}{2}f'\left(i_\mu\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} - \Omega_{\alpha\beta} \wedge i_\mu * \Omega^{\alpha\beta}\right) + \tfrac{1}{2}(f'\mathcal{K} - f) * \theta_\mu + D\lambda_\mu \right\}. \tag{78}$$

The field equations $\delta\mathcal{L}_{tot}^{(n)}/\delta\omega_{\alpha\beta} = 0$ can be solved for the Lagrange multipliers λ^α using (16) with $\Pi^{\alpha\beta} = D(f' * \Omega^{\alpha\beta})$. Therefore, the metric field equations subject to the condition that $\Theta^\alpha = 0$, can be written as

$$4Di_\alpha D(f' * \Omega^\alpha{}_\mu) - \theta_\mu \wedge Di_\alpha i_\beta D(f' * \Omega^{\alpha\beta}) + f' * T_\alpha[\Omega] + \tfrac{1}{2}(f'\mathcal{K} - f) * \theta_\mu = 0, \tag{79}$$

where the canonical energy momentum tensor for the curvature 2-forms,

$$* T_\mu[\Omega] = -\tfrac{1}{2}\left(i_\mu\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} - \Omega_{\alpha\beta} \wedge i_\mu * \Omega^{\alpha\beta}\right), \tag{80}$$

are introduced for convenience. Note that this metric equation holds in $n \geqslant 3$ dimensions. For $f * 1 = \Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta}$, and $f' = 1$, the field equations (79) reduce to

$$Di_\alpha D * \Omega^{\alpha\mu} - \tfrac{1}{4}\theta^\mu \wedge Di_\alpha i_\beta D * \Omega^{\alpha\beta} + *T^\mu[\Omega] = 0, \tag{81}$$

which we shall make use of in the following sections. The trace of Eqns. (81) leads to the following equation

$$(n-1)d*dR + (n-4)\Omega^{\alpha\beta} \wedge *\Omega_{\alpha\beta} = 0, \tag{82}$$

using the Bianchi identity $D*G^{\alpha} = 0$. As for Ricci-squared and $f(R) = R^2$ models, Eqn. (82) depends of the dimensions of spacetime where $n = 4$ appears to be special.

The method adopted can be used to calculate the field equations easily for more complicated-looking actions compared to coordinate methods. As an illustration of this point consider generalized gravitational Lagrangian density in curvature scalar, for example, of the form $f(\mathcal{G}) = \mathcal{G}^2$,

$$\mathcal{G}^2 = \mathcal{K}^2 + 16\mathcal{P}^2 + R^4 - 8R^2\mathcal{P} - 8\mathcal{K}\mathcal{P} + 2\mathcal{K}R^2, \tag{83}$$

contains cross terms of the form $R^2\mathcal{K}$, $R^2\mathcal{K}$ and $\mathcal{K}\mathcal{P}$. The variational derivatives of these types of individual cross terms as well as the models based on the generalized Lagrangian of type, for example $f(R, \mathcal{G})$, are of some interest. In order to illustrate how the method we adopted can easily be used to calculate the variational derivatives of Lagrangian densities in curvature polynomials, consider the generalized Lagrangian density of the form

$$\mathcal{L}_{R^2\mathcal{K}}^{(n)} = R^2 R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} * 1. \tag{84}$$

As before we constrain the variation by adding $\mathcal{L}_{LM}$ to the corresponding action and perform a total variation

$$\delta\mathcal{L}_{R^2\mathcal{K}}^{(n)} = 2R\mathcal{K}\delta\mathcal{L}_{EH}^{(n)} + R^2\delta(\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta}) + \delta\mathcal{L}_{LM}. \tag{85}$$

This explicitly gives

$$\delta\mathcal{L}_{R^2\mathcal{K}}^{(n)} = \delta\omega_{\alpha\beta} \wedge \left\{ D\left\{ (\mathcal{K}R - R^2) * \theta^{\alpha\beta} \right\} - \tfrac{1}{2}(\theta^{\alpha} \wedge \lambda^{\beta} - \theta^{\beta} \wedge \lambda^{\alpha}) \right\} + \delta\lambda_{\alpha} \wedge \Theta^{\alpha}$$
$$+ \delta\theta^{\mu} \wedge \left\{ -\mathcal{K}R * G_{\mu} - R^2 \left(i_{\mu}\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} - \Omega_{\alpha\beta} \wedge i_{\mu} * \Omega^{\alpha\beta} \right) + D\lambda_{\mu} \right\}. \tag{86}$$

Solving $\delta\mathcal{L}_{R^2\mathcal{R}}^{(n)}/\delta\omega_{\alpha\beta} = 0$ for the Lagrange multipliers λ_{α} in terms of other field variables using (16), and using the resulting expression for λ_{μ} in the equation $\delta\mathcal{L}_{R^2\mathcal{R}}^{(n)}/\delta\theta^{\mu} = 0$, the metric field equation becomes

$$-\mathcal{K}R * G^{\alpha} + 2R^2 * T^{\alpha}[\Omega] + D * \left\{ d(R\mathcal{K} - R^2) \wedge \theta^{\alpha} \right\} = 0. \tag{87}$$

As for all the generalized gravitational Lagrangians, written in the above form, field equations has formal resemblance to the field equations (38) and (40) of BD type Scalar-tensor theory where scalar field is replaced by a function of curvature scalars.

C. Weyl Tensor-squared Action

Weyl tensor is the trace-free part of the Riemann curvature tensor and therefore the corresponding Weyl 2-form $C_{\alpha\beta}$ satisfy $C_{\alpha\beta} \wedge *\theta^{\alpha\beta} = 0$. Recall that relative to an orthonormal coframe in $n \geqslant 4$ dimensions curvature 2-form has the expression [27]

$$C^{\alpha\beta} = \Omega^{\alpha\beta} - \tfrac{1}{n-2} \left(\theta^{\alpha} \wedge R^{\beta} - \theta^{\beta} \wedge R^{\alpha} \right) + \tfrac{1}{(n-1)(n-2)} R\,\theta^{\alpha\beta}. \tag{88}$$

It is conformally invariant, has the same symmetry properties with the Riemann tensor and exists in spacetime dimensions $n \geqslant 4$. There is a unique action based on the contraction of the Weyl 2-form with itself

$$\mathcal{L}_W^{(n)} = C_{\alpha\beta} \wedge *C^{\alpha\beta}$$
$$= \Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} - \tfrac{2}{n-2} R^{\alpha} \wedge *R_{\alpha} + \tfrac{1}{(n-1)(n-2)} R^2 * 1. \tag{89}$$

In four dimensional case, Lagrangian density (89) is locally scale invariant and the metric field equations derived from $\mathcal{L}_W^{(4)}$ are known as Bach-Weyl equations which have 4th order partial derivatives of the metric tensor relative to a coordinate basis [49].

As we have mentioned above, the GB Lagrangian density $\mathcal{L}_{\mathcal{G}}^{(4)}$ (55) is a total differential. Thus, without altering the equations of motion for $\mathcal{L}_W^{(4)}$, by adding the term $\mathcal{L}_{GB}^{(4)}$ to the Lagrangian density $\mathcal{L}_W^{(4)}$, the term $\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta}$ in the

Lagrangian density $\mathcal{L}_W^{(4)}$ can be eliminated. This gives somewhat simplified four dimensional gravitational Lagrangian density

$$\mathcal{L}_W^{(4)} = R^\alpha \wedge *R_\alpha - \tfrac{1}{3}R^2 * 1, \tag{90}$$

up to a total differential and an overall multiplicative constant. This reduction of the Lagrangian density makes it possible to derive the field equations of the model $\mathcal{L}_W^{(4)}$ from those of of $f(R) = R^2$ and the Ricci-squared model studied above. This requires the use of the properties and the identities satisfied by the curvature 2-form and therefore the cases of the three and higher dimensions has to be handled separately. Returning to the general $(n \geqslant 4)$ case and following the preceding examples of generalized Lagrangian densities, it is convenient to define 0-form $\mathcal{C} = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ and similarly the generalization of $\mathcal{L}_W^{(n)}$ is assumed to have the form

$$\mathcal{L}_{GW}^{(n)} = f(\mathcal{C}) * 1, \tag{91}$$

where f is an algebraic function of the scalar $\mathcal{C}$. Note that the local scale invariance of the original Lagrangian has been lost in such a generalization. We will study the metric field equations with torsion constrained to vanish and consider the total Lagrangian density $\mathcal{L}_{tot}^{(n)} = \mathcal{L}_{GW}^{(n)} + \mathcal{L}_{LM}$. The variational derivative, in the light of the variational relation given (12), is then given by

$$\delta\mathcal{L}_{tot}^{(n)} = f'\delta\mathcal{L}_W^{(n)} + \delta\theta^\alpha \wedge (f - f'\mathcal{C}) * \theta_\alpha + \delta\mathcal{L}_{LM}, \tag{92}$$

where $'$ in this case denotes derivative with respect to the scalar $\mathcal{C}$. Carrying out the variations indicated in this equations one finds

$$\delta\mathcal{L}_{tot}^{(n)} = \delta\theta^\alpha \wedge \left\{ \frac{4f'}{n-2}R_\beta \wedge *C^{\beta\alpha} + f' * T^\alpha[W] + (f - f'\mathcal{C}) * \theta^\alpha + D\lambda_\alpha \right\}$$
$$+ \delta\omega_{\alpha\beta} \wedge \left\{ 2D(f' * C^{\alpha\beta}) - \tfrac{1}{2}(\theta^\alpha \wedge \lambda^\beta - \theta^\beta \wedge \lambda^\alpha) \right\} + \delta\lambda_\alpha \wedge \Theta^\alpha, \tag{93}$$

where the definition of canonical energy momentum tensor for the Weyl 2-form

$$* T_\alpha[W] = -(i_\alpha C_{\mu\nu}) \wedge *C^{\mu\nu} + C_{\mu\nu} \wedge i_\alpha * C^{\mu\nu}, \tag{94}$$

has been used for convenience and the irrelevant total exterior derivatives are omitted. The field equations $\delta\mathcal{L}_{tot}^{(n)}/\delta\omega_{\alpha\beta} = 0$ can be solved for the Lagrange multipliers λ^α using (16) with $\Pi^{\alpha\beta} = D(f' * C^{\alpha\beta})$. Therefore, the metric field equations subject to the condition that the torsion vanishes, becomes

$$f'\left\{ \frac{4}{(n-2)}R_\beta \wedge *C^{\beta\alpha} + *T^\alpha[W] + 4Di_\beta D * C^{\beta\alpha} + \theta^\alpha \wedge Di_\mu i_\nu D * C^{\mu\nu} \right\}$$
$$+ (f - f'\mathcal{C}) * \theta^\alpha + Di_\beta(df' \wedge *C^{\beta\alpha}) = 0. \tag{95}$$

As a result, the effect of algebraic generalization of the Weyl tensor-squared action is (i) the overall f' factor for the first term and (ii) the last two improvement terms in (95). As expected for consistency of the field equations, for $f(\mathcal{C}) = \mathcal{C}$ the field equations (95) reduce to Bach-Weyl equation [49, 58]. As before, the Palatini variational derivative also lead to field equations that accommodate torsional connection for Weyl-squared action [58]. Recently, in $n > 4$ dimensions, the corrections to Tangherlini solution [59] arising from higher order curvature terms, namely the terms of the form $\mathcal{C} * 1$ are studied in [60] and in this context, it is possible to consider corrections originating from the terms of the form $f(\mathcal{C}) * 1$ in four dimensions since in four dimensions the Schwarzschild solution is not modified by lowest quadratic curvature corrections of the form $R_\alpha \wedge *R^\alpha$ or $R^2 * 1$ [60].

D. Actions Involving Derivatives of Curvature Tensor

In this section we briefly discuss the gravitational Lagrangians involving derivatives of curvature invariants of type, for example $R\Delta R$ [61], where $\Delta = dd^\dagger + d^\dagger d$ is the Laplace-Beltrami operator. The codifferential $d^\dagger$ is the metric dual of the exterior derivative d and we shall use the definition $d* = (-1)^{p+1} * d^\dagger$ acting on p-forms. The models based on this type of Lagrangians as well as all of their algebraic generalizations considered above lead to the metric field equations which are 6th order partial derivatives of metric components relative to a natural coframe. The field equations for such theories involving derivatives of scalar curvature terms have found applications in cosmology [62, 63].

The calculations below can be extended to the generalized Lagrangians of type $dR_{\alpha\beta} \wedge *dR^{\alpha\beta}$ and $dR_{\alpha\beta\mu\nu} \wedge *dR^{\alpha\beta\mu\nu}$ and also to their polynomial generalizations in a straightforward manner.

We shall consider the simplest possible gravitational Lagrangian density involving derivatives of the curvature tensor, namely $R\Delta R$ [61] and its algebraic generalizations. In order to apply the variational procedure used above for generalized gravitational actions presented, it is first convenient to rewrite the action density $R\Delta R * 1$ in the following form

$$\mathcal{L}_{\partial R}^{(n)} = \tfrac{1}{2} dR \wedge *dR = -\tfrac{1}{2}(\partial R)^2 * 1, \tag{96}$$

equivalent to the expression $R\Delta R * 1$ up to an omitted total derivative and an overall sign. We define the scalar $(\partial R)^2$ which takes the form $(\partial R)^2 = g^{\alpha\beta}(\partial_\alpha R)(\partial_\beta R)$ relative to a coordinate frame. This definition will be convenient in the subsequent generalization of the Lagrangian (96) below. The derivation of higher order field equations for gravitational Lagrangians of the form $R\Delta^m R * 1$ is presented in [63].

As before, we constrain the total variation so that the torsion vanishes by adding $\mathcal{L}_{LM}$ to the original Lagrangian density (96) as

$$\mathcal{L}_{tot}^{(n)} = \tfrac{1}{2} dR \wedge *dR + \mathcal{L}_{LM}. \tag{97}$$

We found the total variational derivative of the total Lagrangian density to be

$$\delta\mathcal{L}_{tot}^{(n)} = \delta\omega_{\alpha\beta} \wedge \left\{ 2D((\Delta R) * \theta^{\alpha\beta}) - \tfrac{1}{2}(\theta^\alpha \wedge \lambda^\beta - \theta^\beta \wedge \lambda^\alpha) \right\} + \delta\lambda_\alpha \wedge \Theta^\alpha$$
$$+ \delta\theta_\alpha \wedge \left\{ (\Delta R)\Omega_{\mu\nu} \wedge *\theta^{\mu\nu\alpha} - R\Delta R * \theta^\alpha + *T^\alpha[R] + D\lambda^\alpha \right\}, \tag{98}$$

up to a closed form and the canonical energy momentum $(n-1)$-forms for the scalar curvature R energy momentum forms of a scalar field,

$$* T_\alpha[R] = -\tfrac{1}{2}\{(i_\alpha dR) * dR + dR \wedge i_\alpha * dR\}, \tag{99}$$

has been used for convenience. The canonical energy momentum form of the scalar curvature results from the commutation of variational derivative of the basis coframe 1-form with the Hodge dual exactly as in the case of scalar field (cf. Eqn (39)). The resulting metric field equations [61, 62], subject to the condition that the torsion vanishes, are given by

$$(\Delta R) * R^\alpha + \tfrac{1}{2} * T^\alpha[R] - D * \{(\Delta dR) \wedge \theta^\alpha\} = 0. \tag{100}$$

These equations are 6th order in the components of metric tensor since λ_α turns out to have terms which are 5th order in the partial derivatives of the metric components. The trace of the metric field equations is given by

$$* (\Delta + R)\Delta R - \tfrac{(n-2)}{4(n-1)} dR \wedge *dR = 0. \tag{101}$$

Both (100) and (101) have also formal resemblance to field equations of a BD type scalar-tensor theory (38) and (40) respectively. If matter fields couple minimally to gravitational action (96), then the "scalar field" ΔR act as a variable coupling constant for the metric tensor. In this case, the trace of matter energy momentum fields acts as a source term for the wave equation for the scalar ΔR in (101).

As in the case for the $f(R)$ theory studied above, for the sixth order theory based on (96), the Palatini variational derivative lead to a metric field equation different then (100) as well as it induces a torsion for the connection. In this case, the connection 1-form can be found by solving the relation

$$D((\Delta R) * \theta^{\alpha\beta}) = 0, \tag{102}$$

which lead to nonvanishing, algebraic torsion similar to the case for Riemann-Cartan type $f(R)$ theory.

Finally, we will consider the algebraic generalization of the sixth order theory based on (96). The model then has the Lagrangian density of the form

$$\mathcal{L}_{G\partial R}^{(n)} = f((\partial R)^2) * 1, \tag{103}$$

where, as in all the cases before, f is assumed to be an algebraic function of the scalar $(\partial R)^2$. The total variation of the total Lagrangian density

$$\mathcal{L}_{tot}^{(n)} = \mathcal{L}_{G\partial R}^{(n)} + \mathcal{L}_{LM}, \tag{104}$$

with the $\Theta^\alpha = 0$ condition imposed by Lagrange multiplier term can be found using (12). This gives

$$\delta\mathcal{L}_{tot}^{(n)} = f'\delta\mathcal{L}_{\partial R}^{(n)} - \delta\theta_\alpha \wedge \{(\partial R)^2 f' - f\} * \theta^\alpha + \delta\mathcal{L}_{LM}. \tag{105}$$

Using the total variation (98), the field equations that follow from the expression (105) is

$$f'(\Delta R) * R^\alpha - \tfrac{1}{2}f' * T^\alpha[R] - D * \{f'(\Delta dR) \wedge \theta^\alpha\} + \tfrac{1}{2}\{(\partial R)^2 f' - f\} * \theta^\alpha = 0, \tag{106}$$

which is similar to the field equations based on the model (96) except for the the last terms in curly brackets and the new factor f' as the coefficient of the Ricci 1-form. $'$ denotes derivative with respect to $(\partial R)^2$. The generalized metric equations (106) are 6th order in metric components which is of the same order as the model based on (96). It is interesting to note that, in contrast to the fourth order models, the model derived from the generalized gravitational Lagrangian density $\mathcal{L}_G^{(4)} = \mathcal{L}_{EH}^{(4)} + aR^2 * 1 + bdR \wedge *dR$ is studied in [62] and is shown to be conformally equivalent to the conventional General Relativity coupled to two interacting scalar fields.

V. GRAVITATIONAL CHERN-SIMONS TERM IN FOUR DIMENSIONS

The generalized gravitational actions studied in previous sections indicate that any algebraic function of curvature invariants at the level of action behaves like non-minimally coupled scalar field in the corresponding field equations. In this section, in the light of this important observation, we find the field equations for gravitational action based on the generalization of gravitational Chern-Simons terms [64–66]. In $4n$ dimensions the Chern-Simons form in terms of 1-form-valued matrix notation can be written as

$$Tr[\Omega^{2n}] \equiv \Omega^\alpha{}_\beta \wedge \Omega^\beta{}_\mu \wedge \cdots \wedge \Omega^\sigma{}_\nu \wedge \Omega^\nu{}_\alpha = dK, \tag{107}$$

where Tr denotes trace and $(4n-1)$-form K can be written in terms of connection 1-form as

$$K = 2n \int_0^1 dt\, t^{(2n-1)} Tr[\omega(d\omega + t\omega^2)^{(2n-1)}], \tag{108}$$

as a result of the fact that $Tr[\omega^{4n}] = 0$ [67]. The indices and the wedge products are suppressed for simplicity of the notation. We specialize to four dimensions, namely $n = 1$, and in particular to the Einstein-Hilbert gravitational Lagrangian density modified by parity-violating Pontryagin density, namely $\Omega_{\alpha\beta} \wedge \Omega^{\alpha\beta}$. It is well-known that it is an exact differential, namely,

$$Tr[\Omega^2] = \Omega_{\alpha\beta} \wedge \Omega^{\alpha\beta} = d(\omega_{\alpha\beta} \wedge d\omega^{\alpha\beta} + \tfrac{2}{3}\omega_{\alpha\mu} \wedge \omega^\mu{}_\nu \wedge \omega^{\nu\alpha}), \tag{109}$$

and therefore it does not yield any field equations as it stands. However, in the model of Jackiw-Pi [65], which is based on the Lagrangian density

$$\mathcal{L}_{JP}^{(4)} = \tfrac{1}{2}\Omega_{\alpha\beta} \wedge *\theta^{\alpha\beta} + \theta\,\Omega_{\alpha\beta} \wedge \Omega^{\alpha\beta}, \tag{110}$$

where the $\Omega_{\alpha\beta} \wedge \Omega^{\alpha\beta}$ term couples to a dynamical scalar field θ, it does contribute to the Einstein field equations. The scalar field θ can be regarded as Lagrange multiplier 0-form imposing the constraint $\Omega_{\alpha\beta} \wedge \Omega^{\alpha\beta} = 0$ on the resulting metric field equations,

$$- *G^\alpha + 4Di_\beta(d\theta \wedge \Omega^{\beta\alpha}) - \theta^\alpha \wedge Di_\mu i_\nu(d\theta \wedge \Omega^{\mu\nu}) = 0, \tag{111}$$

which can be found by using (15) and (16). Since the gravitational Chern-Simons term does not contain Hodge dual operator, it does not contribute to the coframe field equation directly but through Lagrange multiplier term in (15). Compared to Lagrange multiplier terms in the field equations for actions containing quadratic curvature invariants studied above, the resulting metric field equations are 3rd order partial differential equations for the Chern-Simons modified theory. Using the general formula (17) together with the first Bianchi identity and $\Omega_{\alpha\beta} + \Omega_{\beta\alpha} = 0$, it is possible to show that neither of the Lagrange multiplier terms on the right hand side of (111) has non-zero trace and therefore the vacuum field equations require $R = 0$.

First order theory based on the Chern-Simons modified gravity (110), with independent coframe and connection 1-forms has also been studied recently and leads to dynamical torsion [68].

In the context of the generalization of gravitational actions we consider, instead of coupling to a non geometrical so-called cosmological scalar field θ it is possible to introduce a term of type $f(\mathcal{T}) * 1$ where $*\mathcal{T} = \Omega_{\alpha\beta} \wedge \Omega^{\alpha\beta}$ is defined

for convenience. Therefore, within the same framework used for the other modified models above, it is natural to consider the Chern-Simons modified Lagrangian density of the form

$$\mathcal{L}_{GJP}^{(4)} = \tfrac{1}{2}\Omega_{\alpha\beta} \wedge *\theta^{\alpha\beta} + f(\mathcal{T}) * 1, \tag{112}$$

where $f = f(\mathcal{T})$ is an algebraic function of the topological invariant $\mathcal{T}$.

In order to find the metric field equations for $\mathcal{L}_{GJP}$, we introduce as for $\mathcal{L}_{JP}$, the total Lagrangian $\mathcal{L}_{tot}^{(4)} = \mathcal{L}_{GJP} + \mathcal{L}_{LM}$. Using (12), the total variational derivative of the total Lagrangian density can be found as

$$\delta\mathcal{L}_{tot}^{(4)} = \delta\omega_{\alpha\beta} \wedge \left\{ 2D(f'\Omega^{\alpha\beta}) - \tfrac{1}{2}(\theta^\alpha \wedge \lambda^\beta - \theta^\beta \wedge \lambda^\alpha) \right\} + \delta\lambda_\alpha \wedge \Theta^\alpha$$
$$+ \delta\theta^\alpha \wedge \left\{ \tfrac{1}{2}\Omega^{\mu\nu} \wedge *\theta_{\alpha\mu\nu} + (f'\mathcal{T} - f) * \theta_\alpha + D\lambda_\alpha \right\}. \tag{113}$$

Solving for the Lagrange multiplier 2-forms in terms of the other fields and then inserting them into the equations for coframe 1-forms we obtain the metric field equations as

$$* G^\alpha = (f'\mathcal{T} - f) * \theta^\alpha + 4Di_\beta(df' \wedge \Omega^{\beta\alpha}) - \theta^\alpha \wedge Di_\mu i_\nu(df' \wedge \Omega^{\mu\nu}), \tag{114}$$

with the condition that $\Theta^\alpha = 0$. In the generalization considered here, the metric field equations (114) remain to be 3rd order in metric components. Note however that, for the simple choice $f(\mathcal{T}) = \mathcal{T}^m$, the first term on the right hand side is like a cosmological constant proportional to $\mathcal{T}^m$ and the Lagrange multiplier part, or $D\lambda^\alpha$ part, contains $d\mathcal{T}$ instead of the gradient of non-geometrical scalar field θ. In contrast to the equations of motion of the gravitational model of Jackiw-Pi (111), the first term on the right hand side appears to be an improvement term. Because of this crucial term, the zero trace ($R = 0$) condition for the vacuum metric equations and the original topological constraint $*\mathcal{T} = \Omega^{\alpha\beta} \wedge \Omega_{\alpha\beta} = 0$ for the original metric equations for $\mathcal{L}_{JP}$ are to be replaced with

$$R = 4(f - f'\mathcal{T}). \tag{115}$$

This equation can be found by tracing the first term on the right hand side in (114), where the Cotton part does not contribute to the trace as in the original Chern-Simons gravitational model of Jackiw-Pi. The new "constraint equation" (115) allows non-zero values of both of the scalars $\mathcal{T}$ and R. The scalar $\mathcal{T}$ can be expressed in terms of the components of the Weyl tensor relative to the complex null coframe of the standard Newman-Penrose formalism. Explicitly in terms of the complex Weyl spinors Ψ_k with $k = 0, 1, 2, 3, 4$, the scalar $\mathcal{T}$ can be written out as

$$\mathcal{T} = 4i \left\{ 3(\bar{\Psi}_2^2 - \Psi_2^2) + (\bar{\Psi}_0\bar{\Psi}_4 - \Psi_0\Psi_4) + 4(\Psi_1\Psi_3 - \bar{\Psi}_1\bar{\Psi}_3) \right\}, \tag{116}$$

where a bar over a quantity denotes complex conjugation. For Petrov type II spaces, it is possible to find a complex null coframe such that $\Psi_0 = \Psi_1 = 0$ with the other Weyl spinors are non-vanishing. Thus, Petrov type II solutions are not allowed in the theory with $\mathcal{T} = 0$. For black hole solutions, which are all of Petrov type D, there is a preferred complex null coframe for which the only non-vanishing Weyl spinor is Ψ_2. In this case, $\mathcal{T} = 0$ requires Ψ_2 to be real and therefore, the Petrov D types with real Ψ_2, such as Schwarzschild space-time, are allowed whereas the Kerr solution with complex Ψ_2 is ruled out. Finally, Petrov types III, N and O (conformally flat) one has $\mathcal{T} = 0$ identically.

VI.   THREE DIMENSIONS

In this last section, we will apply the metric method presented above to the three dimensional cases, in particular to the massive gravity models involving modified gravitational Lagrangian densities. The presentation in the section also reflects the fact that spacetime dimension n is a crucial parameter in modified gravitational models and demonstrates that study of the various modified gravitational actions on a case by case basis can be advantageous.

In three spacetime dimensions the Einstein-Hilbert action (18) does not lead to a dynamical spacetime model [69]. Therefore, various higher order models, such as, third order parity-violating Topologically massive gravity (TMG) [70] theory and New massive gravity (NMG) theory proposed recently [71]. At the linearized level about Minkowski background, NMG can be considered as an extension of Pauli-Fierz theory to massive spin-2 particle in three spacetime dimensions. In a perturbative approach to quantum gravity, NMG has the remarkable properties that it is parity invariant, unitary at tree level, ghost-free [71] and renormalizable [72]. Various classical solutions to NMG theory has been studied. AdS-wave [73], Kundt wave [74] and black hole solutions [75, 76] to NMG has also been worked out. A method, based on Killing vectors, is introduced to study the exact solutions to three dimensional massive gravity theories [77]. In this section we study the field equations of NMG theory and express it in a new form in terms of Cotton 2-form. We will work with General massive gravity (GMG) theory which is based on the combination of TMG

and NMG theories complementing the usual Einstein-Hilbert term [71]. Before we investigate the field equations for General Massive Gravity Theory, we first recall some geometrical identities for the curvature tensors which are special to three spacetime dimensions. In three dimensions, because the Weyl tensor vanishes identically, the Riemanian curvature can be written in terms of contractions and in this case, the identity (88) reduces to the form

$$\Omega^{\alpha\beta} = \theta^\alpha \wedge R^\beta - \theta^\beta \wedge R^\alpha - \tfrac{1}{2}R\theta^{\alpha\beta}. \tag{117}$$

Consequently, the contraction of this identity side by side leads to another useful identity

$$\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} = 2R^\alpha \wedge *R_\alpha - \tfrac{1}{2}R^2 * 1, \tag{118}$$

which will be exploited below. First and the foremost, as a result of the identity (118), in three spacetime dimensions, the most general Lagrangian density involving quadratic curvature terms can be written either in the form

$$aR^\alpha \wedge *R_\alpha + bR^2 * 1, \tag{119}$$

or equivalently in the alternate form

$$a\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} + (2b + \tfrac{1}{2}a)R^2 * 1, \tag{120}$$

with a, b being arbitrary constants. As we have noted before, this is also the case for the gravitational Lagrangians with quadratic curvature terms in four spacetime dimensions as a result of a different geometrical identity (cf. Eqn (65) in four dimensions). As a simple cross-check for the various field equations derived in the previous sections, it is possible to obtain the metric field equations for the Lagrangian density

$$\mathcal{L}_1^{(3)} = \tfrac{1}{2}\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} + \tfrac{1}{4}R^2 * 1, \tag{121}$$

from the metric field equations (68) of the Ricci-squared Lagrangian density

$$\mathcal{L}_2^{(3)} = R^\alpha \wedge *R_\alpha, \tag{122}$$

and compare the resulting equations with those that follow directly from (122). In order to do so, inserting the appropriate expressions from Eqns. (117) and (118) into the metric field equations $*\mathcal{P}_\alpha = 0$ (see Eqn. (68) above), one obtains precisely the metric field equations corresponding to the Lagrangian density (121), which is expected as a result of the equivalence of the Lagrangians provided by the identity in (118) in three spacetime dimensions. These considerations are all in accordance with the consistency of the metric field equations for both of the Lagrangians (121) and (122) which can readily be written down using the general formulae found in the previous sections, namely Eqns. (26), (68) and (81).

NMG also complements the usual Einstein-Hilbert term with a particular combination of the Ricci-squared and scalar curvature-squared terms, which is called K-combination [71]. GMG is an extension of NMG Lagrangian with a Lorentz Chern-Simons term and a cosmological constant Λ, and its original form [71] can be expressed in terms of differential forms as

$$\begin{aligned}\mathcal{L}_{GMG}^{(3)} &= \Lambda * 1 - \tfrac{1}{2}R * 1 + \mathcal{L}_K^{(3)} + \mathcal{L}_{CS}^{(3)} \\ &= \Lambda * 1 - \tfrac{1}{2}R * 1 + \tfrac{1}{2m^2}(R^\alpha \wedge *R_\alpha - \tfrac{3}{8}R^2 * 1) + \tfrac{1}{4\mu}\left(\omega^\alpha{}_\beta \wedge d\omega^\beta{}_\alpha + \tfrac{2}{3}\omega^\alpha{}_\beta \wedge \omega^\beta{}_\gamma \wedge \omega^\gamma{}_\alpha\right),\end{aligned} \tag{123}$$

where μ is the mass parameter in TMG theory whereas m^2 is the mass parameter in NMG theory. Therefore, the GMG Lagrangian, which has two mass parameters, reduces to that of TMG Lagrangian in the limit $m \mapsto \infty$ and to that of NMG Lagrangian in the limit $\mu \mapsto \infty$. Note that the GMG Lagrangian density (123) can equivalently be written in the form

$$\mathcal{L}_{GMG}^{(3)} = \Lambda * 1 - \tfrac{1}{2}R * 1 + \tfrac{1}{4m^2}(\Omega_{\alpha\beta} \wedge *\Omega^{\alpha\beta} - \tfrac{1}{4}R^2 * 1) + \tfrac{1}{4\mu}\left(\omega^\alpha{}_\beta \wedge d\omega^\beta{}_\alpha + \tfrac{2}{3}\omega^\alpha{}_\beta \wedge \omega^\beta{}_\gamma \wedge \omega^\gamma{}_\alpha\right), \tag{124}$$

by using the identity given in (118). The variational derivative of the Lagrangian density (123) (or equivalently, (124)) and the resulting Lagrange multiplier 1-forms have special properties that deserve further scrutiny. We choose to work with the form (124) and the constrained total variation derivative of the Lagrangian density $\mathcal{L}_{tot} = \mathcal{L}_{GMG}^{(3)} + \mathcal{L}_{LM}$ leads to

$$\begin{aligned}\delta\mathcal{L}_{tot} = \delta\omega_{\alpha\beta} \wedge &\left\{\tfrac{1}{2m^2}D * (\Omega^{\alpha\beta} - \tfrac{1}{4}R\theta^{\alpha\beta}) + \tfrac{1}{2\mu}\Omega^{\alpha\beta} - \tfrac{1}{2}D * \theta^{\alpha\beta} - \tfrac{1}{2}(\theta^\alpha \wedge \lambda^\beta - \theta^\beta \wedge \lambda^\alpha)\right\} \\ &+ \delta\theta_\alpha \wedge \left\{(1 + \tfrac{R}{4m^2}) * G^\alpha + (\Lambda + \tfrac{R^2}{16m^2}) * \theta^\alpha + \tfrac{1}{2m^2} * T^\alpha[\Omega] + D\lambda^\alpha\right\} + \delta\lambda_\alpha \wedge \Theta^\alpha,\end{aligned}$$

(up to an omitted closed form) where the definition of the canonical energy momentum 2-form for curvature 2-forms defined in Eqn. (80) has been used for convenience as before. Note here that the trace of the energy momentum 2-form (80) in three dimensions is

$$T^\alpha_{\ \alpha}[\Omega] * 1 = -\tfrac{1}{2}\Omega_{\mu\nu} \wedge *\Omega^{\mu\nu}. \tag{125}$$

The Lagrange multiplier 1-form can be solved in terms of other fields using the field equations obtained from the connection variation using general formulae (16) and (117). This yields

$$\lambda^\beta = \tfrac{1}{m^2}i_\alpha D * (\Omega^{\alpha\beta} - \tfrac{1}{4}R\theta^{\alpha\beta}) + \tfrac{1}{\mu}(R^\beta - \tfrac{1}{4}R\theta^\beta). \tag{126}$$

Here, the first term is the contribution of $\mathcal{L}_K$ terms and are fourth order in metric components, whereas the last term contributes to the metric field equations as the Cotton 2-form $C^\alpha = D(R^\alpha - \tfrac{1}{4}R\theta^\alpha)$ and is third order in metric components. Although Cotton 2-form is defined in arbitrary dimensions and $C_\alpha \otimes \theta^\alpha$ is invariant under conformal transformation, its vanishing is necessary and sufficient condition for conformal flatness in three dimensions. In addition, the Cotton 2-form is traceless, $i_\alpha C^\alpha = 0$, satisfies Bianchi type identity, $\theta_\alpha \wedge C^\alpha = 0$ and C^α is covariantly constant, $DC^\alpha = 0$, in three dimensions [78].

The significance of the particular choice of the Lagrangian density $\mathcal{L}_K$ in (124) shows up in the expression for the Lagrange multiplier terms, namely $D\lambda^\beta$. The first of the terms in (126) is NMG part whereas the second term is the TMG part. The traces of each parts vanishes separately, and therefore they do not contribute to the trace of the metric field equations of the GMG theory, to put it mathematically one has $\theta_\beta \wedge D\lambda^\beta = 0$ for the Lagrange multiplier given in (126). Consequently, as can be explicitly verified from the resulting field equations, the trace of the terms in the field equation coming from the Lagrangian density $\mathcal{L}_K$ add up to $\mathcal{L}_K$ up to a sign. (We refer to the trace formulae given in Eqns. (25), (69) and (82)). Using the result (126), the GMG metric field equations can be written as

$$(1 + \tfrac{1}{4m^2}R) * G^\alpha + (\Lambda + \tfrac{1}{16m^2}R^2) * \theta^\alpha + \tfrac{1}{\mu}C^\alpha + \tfrac{1}{2m^2} * T^\alpha[\Omega]$$
$$- \tfrac{1}{4m^2}D * (dR \wedge \theta^\alpha) + \tfrac{1}{m^2}Di_\beta D * \Omega^{\beta\alpha} = 0. \tag{127}$$

As we shall exemplify below, the metric field equations given in (127) for the GMG field equations seem to be more advantageous compared to the original form of the field equations, since for example, for the fact that it can be also expressible in parts or as a whole in terms of Ricci 1-forms and scalar curvatures using (117). Note that the TMG theory is a third order theory whereas NMG is a fourth order theory. Using (117) and the fact that in three dimensions the Ricci 1-forms and scalar curvature satisfy the identity $i_\alpha DR^\alpha + \tfrac{1}{2}dR = 0$ (This identity can be obtained using the property that Cotton 2-form is traceless) one finds that the contribution of the Lagrange multiplier term, i.e. $D\lambda^\beta$ term, to the metric field equations takes the form

$$D\lambda^\alpha = \tfrac{1}{m^2}D * C^\alpha + \tfrac{1}{\mu}C^\alpha. \tag{128}$$

This form of the Lagrangian multiplier also makes it easier to see the various properties of the NMG theory in comparison to TMG. The first term on the right hand side of (128), which is fourth order in metric components, is unique with the properties: (i) Trace-free, (ii) Covariantly constant, (iii) Parity-preserving. In contrast, the TMG part of the Lagrange multiplier shares these properties except that it is parity-violating.

It is shown that TMG is "square root" of the NMG theory at the linearized level [71]. The presentation of the field equations above makes it possible to sharpen this result. In fact, such a relation between the two theory is encoded in the Lagrange multiplier structure of the GMG theory and can be put in mathematically precise form under certain assumptions using Eqns. (127) and (128). If we assume for simplicity that $\Lambda = 0$ and $R = 0$, then using (127) the TMG and the NMG field equations can be written as

$$(*D + \mu)R^\alpha = 0, \tag{129}$$
$$(*D * D + m^2)R^\alpha = -\tfrac{1}{2}T^\alpha[\Omega], \tag{130}$$

respectively. Eqn. (129) can be derived by simply applying Hodge dual operator $*$ to the TMG equations obtained from (127). The energy momentum 1-form $T^\alpha[\Omega]$ in the right hand side of Eqn. (130) can be expressed in tems of Ricci-squared terms for $R = 0$. This correspondence amounts to the fact that both TMG and NMG can be written in Dirac type and Klein-Gordon type equations respectively. This point of view is elaborated in [79] relative to a coordinate coframe where the correspondence is utilized to provide a method to generate exact solutions for the NMG theory for any given Petrov-Segre type D and N solutions of the TMG theory [69, 78, 80, 81]. In particular, the non-dispersive, line-fronted gravitational wave metric of the Kerr-Schild form

$$g = 2H(u,x)du \otimes du + dv \otimes du + dv \otimes du + dx \otimes dx \tag{131}$$

expressed in terms of local coordinates $\{u, v, x\}$ where u, v are real null coordinates and x is a spatial coordinate. The metric (131) linearizes the TMG equation (129) for the real metric function $H(u, x)$ [82]. Moreover, for the Petrov-Segre type N metric (131), one has $T^\alpha[\Omega] = 0$ identically and therefore, it also linearizes the NMG equations (130). Consequently, there is a class of gravitational wave solutions of the form (131) which is common to both TMG and NMG theories.

VII. CONCLUDING REMARKS

We have presented a unified and practical method to obtain field equations for some modified gravitational Lagrangians containing quadratic curvature invariants and their algebraic generalizations in $n \geqslant 3$ spacetime dimensions. We have adopted the language of differential forms throughout our presentation which provides further insights even for the well-known results. As clearly indicated by our presentation for the GMG theory in three dimensions relative to an orthonormal coframe, the derivation of the metric field equations and the Lagrange multiplier terms provides alternative forms for the field equations and insight for the structure of them.

Although we did not address the question of coupling matter to specific gravitational model, our treatment can be even extended to the case to in which spinor matter fields present or to the case where non-metricity is present. We presented the variational derivatives with respect to independent coframe and connection 1-forms in an explicit form for all the generalized gravitational actions such that these extensions can easily be pursued. The presentation also makes it possible to compare the Palatini formalism with the metric formalism. Rather then providing an exhaustive list of modified gravitational Lagrangians and their corresponding field equations we presented a unified method of the derivation for the field equations using popular gravitational models. We leave the further studies such as the exact solutions, and other physical and mathematical features of the new models we put forward, i.e., modified Chern-Simons theory and GMG theory to forthcoming research.

Appendix

In this section we will briefly explain the notation. For further details we refer to, for example, [27] and [47]. The definitions of basic geometrical quantities we use can be summarized as follows. The metric relative to an orthonormal coframe 1-forms θ^α is $g = \eta_{\alpha\beta}\theta^\alpha \otimes \theta^\beta$, the wedge products of basis 1-forms are abbreviated as $\theta^{\alpha\beta\cdots} = \theta^\alpha \wedge \theta^\beta \wedge \cdots$. The contraction with respect to frame fields e_α is denoted by i_α where $i_\alpha \theta^\beta = \delta^\alpha_\beta$. Oriented volume element can be written as $*1 = \theta^{01\cdots(n-1)}$ where $*$ is the linear Hodge dual operator and $** = (-1)^{s+p(n-p)}$ acting on a p-form in an n dimensional spacetime (s is the metric signature). The indices of the permutation symbol $\varepsilon_{\alpha_1\alpha_2\cdots\alpha_n}$ relative to an orthonormal basis is raised and lowered by $\eta_{\alpha\beta} = \eta^{\alpha\beta}$ and $\varepsilon_{0123\cdots(n-1)} = +1$. Cartan's first structure equations can be used to define torsion 2-form and torsion tensor as

$$\Theta^\alpha = \tfrac{1}{2}T^\alpha{}_{\mu\nu}\theta^{\mu\nu} = D\theta^\alpha = d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta,$$

where $\omega^\alpha{}_\beta = \omega_\lambda{}^\alpha{}_\beta \theta^\lambda$ are connection 1-forms. The contorsion 1-form is defined as $\Theta^\alpha = K^\alpha{}_\beta \wedge \theta^\beta$. The curvature 2-forms in terms of Riemann tensor are

$$\Omega^\alpha{}_\beta = \tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}\theta^{\mu\nu} = d\omega^\alpha{}_\beta + \omega^\alpha{}_\lambda \wedge \omega^\lambda{}_\beta$$

(Cartan's second structure equations). Ricci 1-forms can be defined as the contraction of the curvature 2-forms

$$R_\beta \equiv i_\alpha \Omega^\alpha{}_\beta = \tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}i_\alpha(\theta^\mu \wedge \theta^\nu) = R^\mu{}_{\beta\mu\lambda}\theta^\lambda = R_{\beta\lambda}\theta^\lambda$$

where $R_{\beta\lambda}$ are the components of the Ricci tensor. The scalar curvature is $R = i_\alpha R^\alpha$. Similarly, the components of the Weyl 2-form $C^\alpha{}_\beta$ and those of the Weyl tensor $C^\alpha{}_{\beta\mu\nu}$ are related by $C^\alpha{}_\beta = \tfrac{1}{2}C^\alpha{}_{\beta\mu\nu}\theta^{\mu\nu}$.

A tensor ϕ of type $(r + p, s)$ which is totally antisymmetric with respect to p number of indices can uniquely be associated with the (r, s) tensor-valued p-form $\phi^{\alpha_1\alpha_2\cdots\alpha_s}{}_{\beta_1\beta_2\cdots\beta_r}$. Relative to an arbitrary coframe considered as multilinear maps on (co)frame fields they are related by

$$\phi^{\alpha_1\alpha_2\cdots\alpha_s}{}_{\beta_1\beta_2\cdots\beta_r}(e_{\beta_1}, e_{\beta_2}, \cdots, e_{\beta_p}) \equiv \phi(\theta^{\alpha_1}, \theta^{\alpha_2}, \cdots, \theta^{\alpha_s}, e_{\beta_1}, e_{\beta_2}, \cdots, e_{\beta_r}, e_{\beta_1}, e_{\beta_2}, \cdots e_{\beta_p}).$$

Covariant exterior derivative D acts on tensor-valued p-forms to yield tensor-valued $(p + 1)$-form preserving the tensorial type as follows

$$D\phi^{\alpha_1\alpha_2\cdots\alpha_s}{}_{\beta_1\beta_2\cdots\beta_r} = d\phi^{\alpha_1\alpha_2\cdots\alpha_s}{}_{\beta_1\beta_2\cdots\beta_r} + \omega^{\alpha_1}{}_\lambda \wedge \phi^{\lambda\alpha_2\cdots\alpha_s}{}_{\beta_1\beta_2\cdots\beta_r} + \cdots - \omega^\lambda{}_{\beta_1} \wedge \phi^{\alpha_1\alpha_2\cdots\alpha_s}{}_{\lambda\beta_2\cdots\beta_r} - \cdots$$

More generally, for tensor valued p-form ϕ^I and tensor valued q-form ψ^J

$$D(\phi^I \wedge \psi^J) = D\phi^I \wedge \psi^J + (-1)^p \phi^I \wedge D\psi^J.$$

Acting on scalars D reduces to exterior derivative d. The expression $D\phi^{\alpha_1\alpha_2\cdots\alpha_s}{}_{\beta_1\beta_2\cdots\beta_r}$ can easily be related the covariant derivative $\nabla_X \phi$ using the relation between the exterior and Riemannian covariant derivatives, namely $d = \theta^\alpha \wedge \nabla_\alpha$. The covariant derivative $\nabla_X = X^\alpha \nabla_\alpha$ can be defined by its action on basis coframe: $\nabla_X \theta^\alpha = -(i_X \omega^\alpha{}_\beta)\theta^\beta$.

---

[1] C W Misner, K S Thorne and J A Wheeler, *Gravitation*, W. H. Freeman Publishing, (1973).
[2] E Kretschmann, Annalen der Physik **53**, 575 (1917).
[3] H Weyl, Ann. Phys. **59**, (1919) 101.
[4] A S Eddington, The Mathematical Theory of Relativity, Cambridge University Press, Cambridge, (1923)
[5] T Kaluza, Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl. **1**, 966 (1921); O Klein, Z. Phys. **37**, 895 (1926).
[6] M B Green, J H Schwarz, and E Witten, *Superstring Theory*, Cambridge University Press, London, (1987).
[7] I L Buchbinder, S D Odintsov and I L Shapiro, Effective Actions in Quantum Gravity IOP Publishing, Bristol, (1992); N D Birrell and P C W Davies, (1982) Quantum Fields in Curved Spacetime (Cambridge University Press, Cambridge); G A Vilkovisky Class. Quantum Grav., **9** (1992) 895.
[8] B S DeWitt, Phys. Rev. **160** (1967) 1113.
[9] R Utiyama and B S DeWitt, J. Math. Phys. **3** (1962) 608-618.
[10] S Perlmutter et al., Astrophys. J. 517, 565 (1999); A G Riess et al., Astron. J. 116, 1009 (1998); A G Riess et al., Astron. J. 117, 707 (1999); M Tegmark et al. [SDSS Collaboration], Phys. Rev. D 69, 103501 (2004);. M Tegmark et al. [SDSS Collaboration], Phys. Rev. D 74, 123507 (2006); D. J. Eisenstein et al. [SDSS Collaboration], Astrophys. J. 633, 560 (2005); W J Percival, S Cole, D J Eisenstein, R C Nichol, J A Peacock, A C Pope and A S Szalay, Mon. Not. Roy. Astron. Soc. 381, 1053 (2007); D N Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 148, 175 (2003); D N Spergel, et al. [WMAP Collaboration], Astrophys. J. Suppl. 170, 377 (2007); E Komatsu et al. [WMAP Collaboration], Astrophys. J. Suppl. 180, 330 (2009).
[11] P G Bergmann, Int. J.Theor.Phys. 1, 25 (1968).
[12] T V Ruzmaikina and A A Ruzmaikin, Zh. Eksp. Teor. Fiz. 57, 680 (1969) [Sov. Phys. - JETP 30, 372 (1970)].
[13] B. N. Breizman, V. Ts. Gurovich and V. P. Sokolov, Zh. Eksp. Teor. Fiz. 59, 288 (1970) [Sov. Phys. - JETP 32, 155 (1971)].
[14] H Buchdahl, Monthly Not. Roy. Astr. Soc., **150**, (1970) 1-8.
[15] A A Starobinsky, Phys. Lett. B 91, 99 (1980).
[16] T P Sotiriou and V Faraoni, Rev. Mod. Phys., **82** (2010) 451;
[17] A De Felice and S Tsujikawa, Living Rev. Relativity **13**, (2010) 3 http://www.livingreviews.org/lrr-2010-3.
[18] L Baojiu, J D Barrow and F D Mota, Phys. Rev. D, **76** (2007), 104047; G J Olmo, Hélios Sanchis-Alepuz, S Tripathi, Phys. Rev. D, **80** (2009), 024013.
[19] L Querella, PhD Thesis, Universitè De Liegè, (1998) gr-qc/9902044.
[20] C Will, Living Rev. Relativity **9**, 3 (2006) http://www.livingreviews.org/lrr-2006-3.
[21] H Buchdahl, Nuovo Cimento, **73** (1962) 842.
[22] F W Hehl, J D McCrea, E W Mielke, Y Ne'eman, Phys. Rept. **258**, (1995) 1-171.
[23] H A Buchdahl, J. Phys. A: Math. Gen., **12** (1979) 1229-1234.
[24] S Kichenassamy, Annals of Physics, **168**. Issue 2, (1986) 404-424.
[25] W Kopczynki, Annals of Physics, **203**, (1990) 308-338.
[26] T Dereli and R Tucker, Class. Quantum Grav., **4** (1987) 791-797.
[27] W E Thirring, A Course in Mathematical Physics: Classical Theory of Fields, Springer, 3rd Edition, (2003).
[28] J L Safko, F Elston, J. Math. Phys., **17** Issue 8, (1976) 1531-1737.
[29] Relative to an orthonormal coframe, a matter field couples to the metric tensor through the Hodge dual operator in matter Lagrangian.
[30] M Abdelwahab, M Carloni and P K S Dunsby, Class. Quantum Grav. **25** (2008) 135002.
[31] F W Hehl, P von der Heyde, and G D Kerlick, Rev. Mod. Phys. **48**, (1976) 393.
[32] T Dereli, R W Tucker, Phys. Lett. **110**B, (1981), 206-210.
[33] T P Sotiriou, Class. Quantum Grav., **26** (2009) 152001; S Capozziello , R Cianci, C Stornaiolo and S Vignolo, Class. Quantum Grav. **24** (2007) 6417.
[34] J O'Hanlon and B Tupper, Nuovo Cimento, B**17** (1972) 305.
[35] S Capozzielo, R Cianci, C Stornaiolo and S Vignolo, Class. Quantum Grav. **24** (2007) 6417-6430.
[36] C Lanczos, Rev. Mod. Phys., **29**, (1957) 337. J R Ray, Nuovo Cimennto, B **25** (1975) 706.
[37] T Dereli, R W Tucker, Mod. Phys. Lett. A**17** (2002) 421-428.
[38] T Koivisto, Class. Quant. Grav., **23** (2006) 4289-4296.
[39] B Li, D F Mota, D Shaw, Class. Quant. Grav., **29** (2009) 055018; G Olmo, Phys. Rev. Lett., **98**, (2007) 061101.
[40] E E Flanagan, Class. Quant. Grav., **21** (2003) 417-426; T P Sotiriou, Class. Quantum Grav., **23** (2006) 5117.
[41] R Schimming H-J Scmidt, NTM Schriftenr. Gesch. Naturw. Tech. Med. **27** (1990) 41-48 [arXiv:gr-qc/0412038v1].
[42] K S Stelle, Phys. Rev. D, **16**, Number4 (1979) 953-959; E Alvarez, Rev Mod. Phys. **61**, No.3 (1989) 561-654.

[43] B Zwiebach, Phys. Lett. **156** B, (1985) 315.

[44] M Reuter and C Wetterrich, Nucl. Phys. B **289** 757 (1987); D J Gross and J H Sloan, Nucl. Phys. B 291, 41 (1987); D J Gross and E Witten, Nucl. Phys. B 277, 1 (1986), K S Stelle, Gen. Rel. Grav. 9, 353 (1978); N H Barth and S M Christensen, Phys. Rev. D **28**, 1876 (1983); A Hindawi, B A Ovrut and D Waldram, Phys. Rev. D **53**, 5583 (1996); A Nunez and S Solganik, Phys. Lett. B 608, 189 (2005); T Chiba, JCAP 0503, 008 (2005); I Antoniadis, J. Rizos and K. Tamvakis, Nucl. Phys. B **415**, 497 (1994); N Boulanger, T Damour, L Gualtieri and M Henneaux, Nucl. Phys. B 597, 127 (2001).

[45] D Lovelock, J. Math. Phys. **12**, (1971) 498.

[46] C Lanczos, Z. Phys. **73**, (1932) 147; Ann. Math. **39**, (1938) 842.

[47] I M Benn R W Tucker, Introducton to spinors and geometry with Applications in Physics, Adam and Hilger, IOP Publishing Ltd., (1987)

[48] B Zumino, Phys. Rep. **132**, (1986) 109.

[49] R Bach, Math. Zeits, **9** (1921), 110.

[50] J Madore, Phys. Lett. A **110**, (1985) 289; Phys. Lett. A **111** 289; (1985) 281; Class. and Quantum Grav. **3** (1986), 361.

[51] F Müller-Hoissen, Phys. Lett. B, **163**, (1985) 106; Class. Quantum Grav., **3**, (1986) 665; Europhys. Lett., **3**, (1987) 1075.

[52] Y Choquet-Bruhat, C. R. Acad. Sci., **I 306**, (1988) 445; Y Choquet-Bruhat, J. Math. Phys., **29**, (1988) 1891.

[53] N Deruelle and J Madore, , Phys. Lett. A **114**, (1986) 185; Modern Phys. Lett. A, **1**, (1986) 237; Phys. Lett. B **186**, (1987) 25.

[54] M Arık and T Dereli, Phys. Lett. B **189**, (1987) 96; Phys. Rev. Lett. **62**, (1989) 5.

[55] N Deruelle and J Madore, gr-qc/0305004

[56] A Borowiec, M Ferraris, M Francaviglia, I Volovich, Class. Quantum Grav., **15**, (1998) 43-55.

[57] C N Yang, Phys. Rev. Lett., **33** (1974) 445-447; G Stephenson, Nuovo Cimento, **9** (1958) 263; C W Kilmister, D L Newman, Proc. Cambridge Phil. Soc. (Math. Phys. Sci), **57** (1961), 851; E W Mileke, A R Maggiolo, Gen. Relativ. Gravit., **37**(5) (2005) 997-1007; E W Mileke, Gen. Relativ. Gravit., **40** (2007) 1311-1325.

[58] T Dereli and R Tucker, J. Phys. A: Gen., **15** (1982) L7-L11; B Fiedler and R Schimming, Rep. Math. Phys., **17** (1980) 15-37.

[59] F R Tangherlini, Nuovo Cimento, **27**, (1963) 636.

[60] V P Frolov and I L Shapiro, Phys. Rev. D, **80** (2009) 044034.

[61] H Buchdahl, Acta Math. **85** (1951) 63.

[62] S Gottlober, H-J Schmidt and A A Starobinsky, Class. Quantum Grav., **7** (1990) 893-900; A L Berkin and K Maeda, Phys. Lett. B **245**, 348 (1990); J-C Hwang, Class. Quantum Grav., **8** (1990) L133-L136.

[63] H-J Schmidt, Class. Quantum Grav., **7** (1990) 1023-1031; M Farhoudi, Gen. Relativ. Gravit. **38** (2006) 1261-1284.

[64] F W Hehl, J D McCrea, and E Mielke, J. Math. Phys., **38**(8) (1991), 2169-2180.

[65] R Jackiw and S Y Pi, Phys. Rev. D, **68** (2003), 104012.

[66] S Alexander and N Yunes, Phys. Rept., **480** (2009) 1.

[67] S S Chern and J Simons, Annals Math. **99** (1974) 48-69.

[68] M B Cantcheff, Phys. Rev. D, **68** (2003), 104012, S Alexander and N Yunes Phys. Rep., **480** (2009) 1.

[69] J D Barrow, A B Burd and D Lancaster, Class. Quantum Grav., **3** (1986) 551-567.

[70] S Deser, R Jackiw, Templeton, Phys. Rev. Lett. **48**, (1982) 975978.

[71] E A Bergshoeff, O Hohm and P K Townsend, Phys. Rev. Lett., **102** (2009) 201301;

[72] I Oda, JHEP, **0905**, (2009) 064.

[73] E Ayón-Beato, G Giribet, and M Hassaïne, JHEP, **0905**, (2009) 029.

[74] M Chakhad, arXiv:0907.1973v1 [hep-th] (2010).

[75] J Oliva, D Tempo, R Troncoso, JHEP 0907:011, (2009); G Clement, Class. Quantum Grav., **26**, (2009) 165002.

[76] E A Bergshoeff, O Hohm and P K Townsend, Phys. Rev. D, **79** (2009) 124042.

[77] M Gürses, arXiv:1001.1039v4 [gr-qc], (2010).

[78] A A García, F W Hehl, C Heinicke, A Macías, Class. Quantum Grav., **21** (2004) 1099-1118.

[79] H Ahmedov, A N Aliev, arXiv:1006.4264v3 [hep-th] (2010); H Ahmedov, A N Aliev, arXiv:1008.0303v1 [hep-th](2010).

[80] G S Hall, T Morgan and Z Perjes, Gen. Relativ. Grav., **19** 1137 (1987); D D K Chow, C N Pope and E Sezgin, Class. Quantum Grav., **27** (2010) 105001.

[81] The Cotton tensor is classified (into Petrov types) according to the eigenvalues of traceless 3×3 matrix $C_{\alpha\beta} = C_{\beta\alpha}$ which can be defined as $(C_{\alpha\beta}) = i_\beta * C_\alpha$ in terms of Cotton 2-form. The TMG field equations render the classification of traceless Ricci tensor (into Segre types) and Petrov type equivalent.

[82] T Dereli, R W Tucker, Class. Quantum Grav., **5**, (1988) 951-959.